Bachelor Thesis

Interval Colorings of Graphs

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Abstract

Discrete mathematics is a flourishing area of mathematics which deals with "discrete structures". Especially graph theory, which lends itself to a variety of different applications such as networks and route planning, is researched extensively.

This thesis is devoted to a particular edge coloring problem, that of interval colorability: A graph G is called interval colorable if there is an assignment of integers to its edges such that each vertex is incident to edges colored by a set of consecutive integers and there are no two adjacent edges of the same color. Motivated by the problem of finding compact school timetables, i.e. timetables such that the lectures of each teacher and each class are scheduled in consecutive periods, Asratian and Kamalian first introduced the notion. However, as not every graph is interval colorable, the notion was extended to the one of interval thickness where we want to edge-decompose the graph into as few interval colorable subgraphs as possible.

First, we survey some results in the study of interval colorability and related topics. In particular, we focus our attention on current results on the interval thickness. After that, among other new results, we will present a new upper bound on the interval thickness using a powerful tool in extremal graph theory and then show that the spectrum, which we define as the set of all positive integers t such that there is an interval coloring using the integers 1 to t, can have arbitrarily many and arbitrarily large gaps. Lastly, a highly-cited, but hardly available paper in the field written by Sevastianov is translated with additional comments.

Die diskrete Mathematik ist ein aufblühendes Teilgebiet der Mathematik, welche sich grob gesagt mit "diskreten Strukturen" befasst. Insbesondere in der Graphentheorie, welche sich in vielen verschiedenen Anwendungen wie Netzwerke oder Routenplanung als wichtiges Werkzeug etabliert hat, wird intensiv geforscht.

Diese Abschlussarbeit befasst sich mit einem speziellen Kantenfärbungsproblem, dem der Intervallfärbbarkeit: Ein Graph G heißt intervallfärbbar, wenn eine Zuweisung von ganzen Zahlen zu den Kanten existiert, sodass die inzidenten Kanten jedes Knotens durch eine Menge auffeinander folgender Zahlen gefärbt wird, und je zwei adjazente Kanten nicht die selbe Farbe haben. Motiviert durch das Problem, einen "kompakten" Stundenplan, in dem die Unterrichtsstunden sowohl für Lehrer als auch Schüler auffeinanderfolgend sind, zu erstellen, haben Asratian und Kamalian diesen Begriff eingeführt. Weil aber nicht jeder Graph intervallfärbbar ist, wurde der Begriff erweitert durch den der Intervalldicke, wo wir einen Graphen in möglichst wenige intervallfärbbare Teilgraphen kantendisjunkt zerlegen wollen.

Zuerst geben wir einen Überblick über einige wichtige Resultate der Intervallfärbbarkeit und verwandte Themen. Dabei fokussieren wir uns insbesondere auf die jetzigen Resultate über die Intervalldicke. Danach präsentieren wir, unter anderem, eine neue obere Schranke für die Intervalldicke mithilfe eines mächtigen Resultats der extremalen Graphentheorie, und zeigen, dass das Spektrum, welches wir definieren als die Menge aller natürlichen Zahlen t, für die unser Graph eine Intervallfärbung mit den Farben 1 bis t besitzt, beliebig viele und beliebig große Lücken haben kann. Zuletzt wird eine hochzitierte, aber schwer erhältliche Arbeit in dem Gebiet, welche von Sevastianov verfasst wurde, übersetzt mit zusätzlichen Kommentaren.

Contents

Chapter 1

Introduction

The study of graph colorings is a very broad and rich area of graph theory. Historically, one of the key problems has been the *Four Color Problem*[1] (see [22, p. 146]), whether every map can be colored with four colors so that adjacent countries are shown in different colors.

We are interested in a question which originated from scheduling theory.

EXAMPLE 1. *Let's say we have n people at a meeting. Certain pairs of these people want to talk to each other where the talks always have the same fixed length. Our job is now to schedule the talks, i.e. each of the talks is assigned a certain time slot, such that for each person the talks are without breaks.*

If we construct the graph G where the set of people correspond to our vertices and the edges correspond to the talks, then by associating the time slots with integers $1, 2 \dots,$ the question in Example 1 becomes equivalent to asking whether it is possible to color the edges of G such that for each vertex (or person) the colors of the incident edges form an interval.

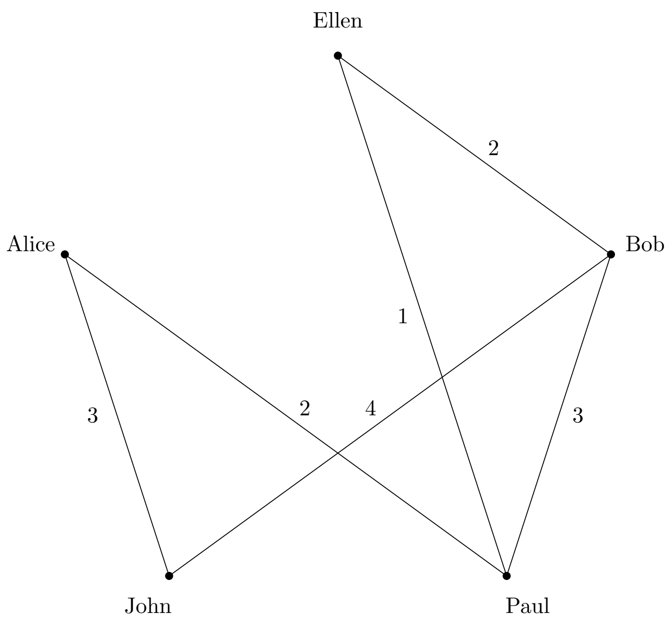

FIGURE 1. Sample graph for Example 1 with 4-interval coloring

[1]Later famously turned into the Four Color Theorem due to Appel and Haken in [3].

Label	Time Slot	Talks
1	(09:00 – 09:30)	Ellen & Paul
2	(09:30 – 10:00)	Bob & Ellen, Alice & Paul
3	(10:00 – 10:30)	Alice & John, Bob & Paul
4	(10:30 – 11:00)	Bob & John

FIGURE 2. Corresponding time table to the 4-interval coloring

Roughly speaking, we will say that a graph is *interval colorable* if such a coloring exists. Naturally, we will also refer to the coloring as an *interval coloring* and more specifically a *t-interval coloring* if the colors $1, \dots, t$ are used.

However, it turns out that rarely such a coloring exists. Thus, people have looked at various ways to relax the conditions of an interval coloring that in a sense still allow for schedulings with minimal waiting time. One such relaxation can be thought of as asking the following question to Example 1:

Can the talks be distributed on k days such that for each person the talks are consecutive on each day?

In more general terms, we are looking for a schedule with k "stages" where in each stage the "no-wait" condition holds, which in [7] is referred to as a "no-wait multi-stage schedule".

In graph theoretic terms, this is equivalent to asking whether the constructed graph can be edge-decomposed into k interval colorable subgraphs. Of course, our goal is then to minimize k. We will call that minimum k the *interval thickness.*

In this thesis, we both survey older results in the study of interval colorability and related topics that cumulated since Asratian and Kamalian first introduced the notion in 1987 (see [9]) and also present some new results:

1. *Consecutive layers of the Boolean lattice are interval colorable.* (Cor. 7)

2. *For every $k, d \in \mathbb{N}$ there exists a planar bipartite graph G for which the spectrum, the set of t for which G is t-interval colorable, has more than k gaps, where each gap is at least d.* (Thm. 49)

3. *For every $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for every $n > n_0$ and every n-vertex graph G has at most interval thickness εn.* (Cor. 10)

The last result in particular is an improvement to current general upper bounds on the interval thickness which all depend linearly on the maximum degree and therefore, for dense graphs[2], linearly on the number of vertices.

We also translate a highly-cited paper by Sevastianov (see [62]) in which he proved that deciding whether a bipartite graph is interval colorable or not is $\mathcal{NP}$-complete. This result was of interest in two ways: Not only does it imply that bipartite graphs, which usually have nice properties, are not in general interval colorable, but also that the problem to decide that is generally hard.

[2]I.e. where the number of edges is in $\Theta(n^2)$ where n is the number of vertices.

Chapter 2

Preliminaries

In this chapter, we define most of the terminology we will use. For all other undefined terms, we refer to [22].

1. General terminology

- For this thesis, we will let $\mathbb{N}$ denote the positive integers, i.e. $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0$ denote the non-negative integers, and $\mathbb{Z}$ denote the integers.

- For integers a, b, let $[a, b] \coloneqq \{z \in \mathbb{Z} \colon a \leq z \leq b\}$. In particular, define $[k] \coloneqq [1, k]$ for $k \in \mathbb{N}$. In general, we will refer to these sets of consecutive integers as *intervals*. The *length* of an interval $[a, b]$ is given by $|[a, b]| = b - a + 1$.

- For a finite set X and $k \in \mathbb{N}$, we define
$$2^X \coloneqq \{X' \subseteq X\} \text{ and } \binom{X}{k} \coloneqq \{S \subseteq X \colon \ |S| = k\}.$$
It is a well-known combinatorial result that $\left|2^X\right| = 2^{|X|}$ and $\left|\binom{X}{k}\right| = \binom{|X|}{k}$.

- For a function $f \colon X \to Y$, we let $f(X) = \{f(x) \mid x \in X\}$ denote the *image* of f and $f\big|_{X'}$ denote the *restriction* of f onto $X' \subseteq X$, i.e. the function
$$f\big|_{X'} \colon X' \to Y, x \mapsto f(x).$$

- All graphs considered in this thesis are *simple*, i.e. of the form $G = (V, E)$, where V is a finite set and $E \subseteq \binom{V}{2}$. We refer to $|G| \coloneqq |V|$ as the *order* of G and $\|G\| \coloneqq |E|$ as the *size* of G.

- We define the *complete graph on n vertices* to be $K_n \coloneqq ([n], \binom{[n]}{2})$ for $n \in \mathbb{N}$. We say that a graph G is *complete* or a *clique* if it's (up to isomorphism) K_n for some $n \in \mathbb{N}$.

- Generally, for $n_1, \ldots, n_r \in \mathbb{N}$ we will let $K_{n_1,\ldots,n_r}$ be the complete r-partite graph with r parts of size $n_1, \ldots, n_r$ respectively. In particular, we let K_{n*r} denote the r-partite graph $K_{\underbrace{n, \ldots, n}_{r \text{ times}}}$.

- We define the *path of length n* to be
$$P_n \coloneqq (\{v_1, \ldots, v_n, v_{n+1}\}, \{v_i v_{i+1} \mid i \in [n]\}$$
for $n \geq 0$. We say that a graph G is a path if it's (up to isomorphism) P_n for some $n \geq 0$.

- We define the *cycle of length* n to be

$$C_n \coloneqq (\{v_1, \ldots, v_n\}, \{v_i v_{i+1} \mid i \in [n-1]\} \cup \{v_n v_1\})$$

for $n \geq 3$. We say that a graph G is a cycle if it's (up to isomorphism) C_n for some $n \geq 3$.

- The *degree* of a vertex v is the number of vertices that are *adjacent* to v, which we denote by $\deg(v)$. For a graph $G = (V, E)$, we denote the minimum and maximum degree as $\delta(G) \coloneqq \min_{v \in V} \deg(v)$ and $\Delta(G) \coloneqq \max_{v \in V} \deg(v)$.

- The *n-dimensional hypercube* Q_n is defined by

$$Q_n \coloneqq (2^{[n]}, \{S, T\} : S, T \in 2^{[n]}, |S \triangle T| = 1)$$

where $S \triangle T = (S \setminus T) \cup (T \setminus S)$ denotes the *symmetric difference.*

- We call a graph *r-regular*, if the degree of every vertex is r. For $r = 1$, we say that the graph is a *matching* and for $r = 3$, we also say that the graph is *cubic* and *subcubic* if the degree of every vertex is at most 3.

- The *distance* between two vertices u, v is defined as the length of the shortest path linking those two vertices and is denoted by $d(u, v)$. The *diameter* $\operatorname{diam}(G)$ of a graph G is the greatest distance between any two vertices.

- We say that a graph is *connected* if any two distinct vertices a, b are linked by a path. We refer to such a path as an *a-b-path* and say that a and b are the path's *endpoints.*

- We say that $H = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V, E' \subseteq E$. We then write $H \subseteq G$. The *union* of G and H is defined by

$$G \cup H' \coloneqq (V \cup V', E \cup E').$$

- For $G = (V, E)$ and $X \subseteq V$, we define the subgraph $G[X] \coloneqq (X, E \cap \binom{X}{2})$.

- For $G = (V, E)$ and $F \subseteq \binom{V}{2}$, we denote $G - F = (V, E \setminus F)$.

- For a fixed graph $G = (V, E)$, we say that $G_1, \ldots, G_k \subseteq G$ is an edge-decomposition if

$$\forall\, i \in [k] \colon V(G_i) = V \text{ and } \dot{\bigcup_{i=1}^{k}} E(G_i) = E.$$

- We say that to graphs G and H are *isomorphic* and write $G \simeq H$ if they are the same graph "up to relabeling" of the vertices.

- A *hypergraph* is given by $\mathcal{G} \coloneqq (X, \mathcal{M})$ where X is a finite *ground set* and $\mathcal{M} \subseteq 2^X$ is the set of *hyperedges.*

2. Edge colorings and interval colorings

In this section, we will first revise some definitions and theorems that specifically deal with edge colorings.

Definition 1 (Edge coloring, k-edge coloring). *Let c be a coloring of the edges of $G = (V, E)$. We say that c is a (proper) edge coloring, if for any pair of adjacent edges $e, e' \in E$ we have $c(e) \neq c(e')$. If $c(E) = [k]$, we say that c is a k-edge coloring.*

One thing of interest both from a theoretical point of view and for applications is to minimize the number of colors in the edge coloring, which is formally captured by the chromatic index.

Definition 2 (Chromatic index). *The chromatic index of G is the smallest $k \in \mathbb{N}$ such that G has a k-edge coloring. It is denoted by $\chi'(G)$.*

While determining the chromatic index of a given graph G is generally hard ($\mathcal{NP}$-hard in fact, see [34]), a tight bound is famously given by *Vizing's Theorem.*

Theorem 1 (Vizing's Theorem, [22, Thm. 5.3.2]). *For any graph G it holds*

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

Definition 3. *A graph G is said to be Class 1 if $\chi'(G) = \Delta(G)$ and Class 2 if $\chi'(G) = \Delta(G) + 1$. By Vizing's Theorem, every graph is in one of these classes.*

There is also famously a positive result for bipartite graphs.

Theorem 2 (Kőnig's Edge Coloring Theorem, [22, Prop. 5.3.1]). *Every bipartite graph is Class 1.*

Building on those definitions, we can now define interval colorability.

Definition 4 (Interval colorability, t-interval colorability). *A graph $G = (V, E)$ is called interval colorable if it has a proper edge coloring $c\colon E \to \mathbb{Z}$ such that the colors of the edges incident to every vertex of G form an interval of integers. Furthermore, if $c(E) = [t]$, we say that G is t-interval colorable.*

Note that, in the context of Example 1 t-interval colorability means that only t time slots are necessary. Generally, we will hence refer to t as the *span* of the coloring.

To keep the proofs compact, we introduce the following definitions.

Definition 5 (Palette). *The palette of a vertex $v \in V$ in a graph $G = (V, E)$, with respect to an edge coloring c of G, is $P_c(v) := \{c(e)\colon e \in E, e \text{ is incident to } v\}$.*

Remark 1. *Note that c is an interval coloring if and only if $P_c(v)$ forms an interval of length $\deg(v)$ for all $v \in V$. In particular, we have $|P_c(v)| = \deg(v)$ for all $v \in V$ if c is a proper edge coloring.*

Definition 6 (Spectrum, $w(G)$, $W(G)$). *For a given graph G, we define*

$$\operatorname{spec}(G) := \{t \in \mathbb{N}_0 \colon G \text{ is } t\text{-interval colorable.}\}$$

as the spectrum of G. In particular, G is interval colorable if and only if $\operatorname{spec}(G) \neq \emptyset$. For interval colorable G, we denote $w(G) := \min \operatorname{spec}(G)$ and $W(G) := \max \operatorname{spec}(G)$.

In the context of schedules like Example 1, $w(G)$ and $W(G)$ may be interpreted as how tight or stretched out the schedule can be chosen. Naturally, we will therefore also refer to $w(G)$ and $W(G)$ as the *minimum* and *maximum span* respectively.

Definition 7 (Symmetric coloring). *For a given graph G and t-edge coloring $c\colon E(G) \mapsto [t]$, the symmetric coloring c' is given by*

$$c'\colon E(G) \mapsto [t], e \mapsto t+1-c(e).$$

Remark 2. *Note that c is a t-interval coloring of G if and only if the symmetric coloring c' is a t-interval coloring.*

Definition 8 (Interval thickness, [7]). *For a graph G, we define the interval thickness of G to be the smallest $k \in \mathbb{N}$ such that G admits an edge-decomposition into k interval colorable subgraphs. It is denoted by $\theta_{\mathrm{int}}(G)$.*

3. Some lemmas

In this section, we present some auxiliary lemmas.

First, we will discuss why our definition of interval colorability (see Definition 4) is "sufficiently strict" despite being more flexible: In practical applications to scheduling, one wants the color labels to be in $\mathbb{N}$ or $\mathbb{N}_0$, or even as in [10] restrict interval colorable graphs to graphs that are t-interval colorable for some $t \in \mathbb{N}_0$, i.e. for which no time slot in the span of the corresponding schedule is completely idle. However, this can be easily guaranteed from our "weaker" definition of interval colorability.

Lemma 1. *Let $G = (V, E)$ be interval colorable and $v \in V$. Then, for any interval $[a, b]$ of length $\deg(v)$ there exists an interval coloring such that the colors of edges incident to v form $[a, b]$. In particular, for any $s \in \mathbb{Z}$, there exists an interval coloring c of G with $s = \min c(E)$.*

Proof. By definition, G possesses an interval coloring c. Let $[a', b']$ be the interval of length $\deg(v)$ formed by the colors of edges incident to v. Then

$$c'(w) := c(w) + (a - a')$$

defines an interval coloring of G, as the "transposition" preserves the consecutiveness of the colors. Furthermore, the set of colors of edges incident to v for c' are by construction $[a' + (a - a'), b' + (a - a')] = [a, b]$ since

$$b' - a' + 1 = |[a', b']| = \deg(v) = |[a, b]| = b - a + 1.$$

Thus, c' is the desired coloring.

Furthermore, if we choose v to be a vertex incident to an edge colored $\min c(E)$ and choose $[a, b] = [s, s + \deg(v) - 1]$, then $\min c'(E) = (\min c(E)) + (s - \min c(E)) = s$. $\square$

Lemma 2. *Let $G = (V, E)$ be a connected, interval colorable graph with interval coloring c. Then $c(E)$ is an interval.*

Proof. We will proceed by induction on $f := |c(E)|$: If $f = 1$, then $c(E)$ is trivially an interval. Assume now that for all interval colorable graphs and all their interval colorings on at most $f \in \mathbb{N}$ colors the image of the coloring forms an interval.

Let $G = (V, E)$ be an interval colorable graph with $(f+1)$-interval coloring c and let $s \coloneqq \min c(E)$ and $S \coloneqq \max c(E)$ denote the smallest and largest color of c respectively. Let C_S be the color class S and G' be a connected component of $G - C_S$ with an edge colored s. As G is connected, G' must contain a vertex v incident to an edge in C_S. By maximality of S and due to $E(G') \neq \emptyset$, v must be incident to an edge colored $S-1$. Therefore, as c restricted onto G' naturally induces an f-interval coloring of G', $c(E(G')) = [s, S-1]$ by induction hypothesis.

Thus, $c(E) = [s, S-1] \cup \{S\} = [s, S]$. □

Corollary 1. *Every interval colorable graph is t-interval colorable for some $t \in \mathbb{N}_0$.*

Proof. Let G be interval colorable and let $G_1, \dots, G_l$ be the connected components of G. Clearly, each of the G_i's $(1 \leq i \leq l)$ is interval colorable. Therefore, using Lemma 1 and Lemma 2, there exists a t_i-interval coloring c_i of G_i for some $t_1, \dots, t_l \in \mathbb{N}_0$. Thus, with $t \coloneqq \max\{t_1, \dots, t_l\}$,

$$c\colon E(G) \to [t], e \mapsto c_i(e) \text{ if } e \in E(G_i), i \in [l]$$

defines a t-interval coloring of G. □

We conclude this section with some simple, but helpful bounds.

Lemma 3. *Let c be an interval coloring of a graph $G = (V, E)$. Then:*

(1) $\max P_c(v) = \min P_c(v) + \deg(v) - 1$,

(2) $k - (\deg(v) - 1) \leq \min P_c(v) \leq \max P_c(v) \leq k + (\deg(v) - 1)$ *if* $k \in P_c(v)$ *for* $v \in V$,

(3) if $P = v_0 \dots v_n$ is a path in an interval colorable graph G, then

$$c(v_0v_1) - \sum_{k=1}^{n-1} (\deg(v_k) - 1) \leq c(v_{n-1}v_n) \leq c(v_0v_1) + \sum_{k=1}^{n-1} (\deg(v_k) - 1).$$

Proof. Trivially, (1) holds. From (1), as $\min P_c(v) \leq k \leq \max P_c(v)$, we also immediately get (2).

For (3), we proceed by induction on n: Obviously, the base case $n = 1$ holds. So, assume that for some $n \in \mathbb{N}$ the claim is true and let $P = v_0 \dots v_n v_{n+1}$. Since v_n is incident to $v_{n-1}v_n$, we get from (2)

$$c(v_{n-1}v_n) - (\deg(v_n) - 1) \leq c(v_n v_{n+1}) \leq c(v_{n-1}v_n) + (\deg(v_n) - 1).$$

Applying our induction hypothesis on Pv_n and plugging in the bound for $c(v_{n-1}v_n)$, we get

$$c(v_0v_1) - \sum_{k=1}^{n} (\deg(v_k) - 1) \leq c(v_n v_{n+1}) \leq c(v_0v_1) + \sum_{k=1}^{n} (\deg(v_k) - 1).$$

Thus, (3) also holds. □

Chapter 3

Interval edge colorings from the perspective of open shop scheduling

As indicated in the introduction, the study of interval colorings is very much motivated by finding schedulings without waiting periods, which we will also refer to as being *compact*. For example, Asratian and Kamalian described in [10] how creating compact school timetables, where neither classes nor lecturers experience any gaps in the schedule, can be modeled using interval colorings of bipartite graphs. Here, we will contextualize the interval coloring problem for bipartite graphs in the framework of *open shop schedules*, as introduced in [47].

Definition 9 (Open shop). *An open shop consists of a set of machines, also called "processors", $\mathcal{M} = \{M_1, \dots, M_m\}$ and a set of jobs $\mathcal{J} = \{J_1, \dots, J_n\}$. Each job J_i is made up of m operations $O_{i,1}, \dots, O_{i,m}$ where $O_{i,h}$ requires machine M_h for its execution. In a feasible schedule, the operations of a job can be executed in any order; however, executions of any two operations of the same job may not be done in parallel. Each machine can execute at most one job at a time. A job completes its execution as soon as all its operations have been executed. The completion time of job J_i in a schedule S is denoted by $C_i(S)$. The processing time of $O_{i,h}$ is denoted by $p_{i,h} \in \mathbb{R}_{\geq 0}$. Hence, we can specify an open shop instance by a non-negative real $n \times m$ matrix*

$$\mathbb{P} = \begin{pmatrix} p_{1,1} & \cdots & p_{1,m} \\ \vdots & \ddots & \vdots \\ p_{n,1} & \cdots & p_{n,m} \end{pmatrix}.$$

For $i \in [n]$, the length of job J_i is defined by

$$P_i = \sum_{h=1}^{m} p_{i,h}.$$

For $h \in [m]$, the workload for machine M_h is defined by

$$L_h = \sum_{i=1}^{n} p_{i,h}.$$

To again emphasize, the difference between open shops and job shops (or flow shops) is that the order in which the operations of a job are executed can be chosen *arbitrarily*, while in job shops and flow shops the order of the operations is fixed. It is also important to note that schedulings "start" by time $t = 0$.

As discussed, interval colorings *model* particular schedules with a "no-wait" condition which we formalize as *compact schedules.*

Definition 10 (Compact schedules). *We say that an open shop schedule is compact if for each job J_i there is a time interval $[S_i, S_i + P_i]$ where the job is processed and* *where for each machine M_h there is a time interval $[s_h, s_h + L_h]$ where the machines process all its operations. In other words, there is no waiting between operations of the job and no idle time between operations on machines.*

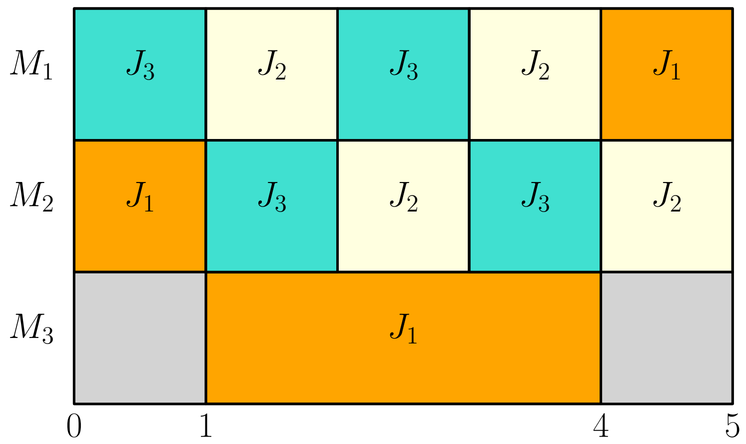

FIGURE 1. Compact schedule for $\mathbb{P} = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & 0 \\ 2 & 2 & 0 \end{pmatrix}$

Note that in general the intervals in the definition are real intervals.

Apart from the examples previously given, such conditions can also "arise from characteristics of the processing technology (e.g. temperature, viscosity) or from the absence of storage capacity between tasks of a job (e.g. lack of buffers)", see [29].

If we now only consider open shops with operations $p_{i,h} \in \{0, 1\}$ ("0-1-operations"), i.e. where the length of a non-zero operation is always the same, then the connection between compact schedules and interval colorings for bipartite graphs becomes immediate: Constructing the bipartite graph G with parts $\mathcal{M}$ and $\mathcal{J}$ and an edge between J_i and M_h if and only if $p_{i,h} = 1$, then finding an interval coloring for G naturally corresponds to a compact schedule for $\mathbb{P}$.

We may also note that often times, as with classical open shop problems, one wants to minimize the *makespan* $C_{\max}(S) := \max_{i \in [n]} C_i(S)$ which corresponds to finding a $w(G)$-interval coloring.

Chapter 4

Survey of previous results and observations

In the past 30 and more years, a lot of papers have been written about this topic. Individual graph classes were studied and different variations on the property introduced. Here, we will now survey some of these results. For that, we will restrict ourselves to (simple) graphs, as defined in chapter 2. However, several results also hold for multigraphs or graphs with loops.

1. General properties of interval colorings

We first start with some more general properties of interval colorings.

1.1. Class 1 vs. Interval colorability. Of course, to see how applicable interval colorings and their corresponding schedulings are, it is important to study which graphs are interval colorable and which aren't. It is easy to see that not all graphs are interval colorable, a triangle for example comes to mind: To edge-color a triangle properly, all colors need to be distinct, so two of the colors can't be consecutive. In general, it is known that in fact none of the graphs in Class 2 are interval colorable.

THEOREM 3 ([10, Prop. 1]). *If G is interval colorable, then G is Class 1.*

PROOF. Let $G = (V, E)$ be an interval colorable graph and let c be an interval coloring of G. Define $c' \colon E \to [\Delta(G)], e \mapsto c(e) \pmod{\Delta(G)}$. Trivially, this new coloring c' uses $\Delta(G)$ colors. It is also clear that c' is a proper coloring: Let $v \in V$ be a vertex and let $P_c(v) = [a, b]$ be the interval of colors of the edges incident to v. As $\deg(v) \leq \Delta(G)$, the interval has at most length $\Delta(G)$. Thus, as the colors are consecutive, they get mapped to different classes of modulo $\Delta(G)$. So, all the colors of edges incident to v are distinct, hence the coloring proper. Thus, G is Class 1. $\square$

However, not all graphs that are Class 1 are also interval colorable.

LEMMA 4. *For $n \in \mathbb{N}$, let F_n denote the n-th friendship graph, i.e. the graph of n K_3-copies which have exactly one vertex v in common and are otherwise pairwise disjoint. Then F_n is interval colorable if and only if $n \equiv 0 \pmod 2$. Furthermore, F_n is Class 1 for all $n \geq 2$.*

PROOF. We first show that F_n is only for $n \equiv 0 \pmod 2$ interval colorable by induction on n:

The base case $n = 1$ is clear and F_2 is interval colorable as we can see in Figure 1.

Now, consider $n \geq 3$ for the induction step. Let's assume that $F := F_n$ is interval colorable with interval coloring c. Let v be the center vertex contained in all n K_3-copies and let $\{a_i, b_i\} \subseteq V$ be the remaining vertices of the i-th K_3-copy for

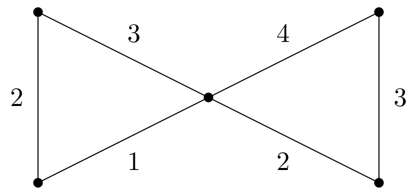

FIGURE 1. Interval coloring of F_2

$i = 1, \dots, n$. As $\deg(v) = 2n$, we may w.l.o.g. assume that the colors of edges incident to v form the interval $[2n]$ by Lemma 1. As c is a proper interval coloring by assumption, we must have

$$\{c(a_i v), c(b_i v)\} = \{c(a_i b_i) - 1, c(a_i b_i) + 1\}$$

for all $i \in [n]$. In particular, the edges incident to v colored $2n-1$ and $2n$ aren't part of the same K_3-copy. Therefore, by suitably relabeling the vertices and permuting the order of the copies, we may assume that $c(b_n v) = 2n$ and $c(b_{n-1} v) = 2n - 1$.

It follows that

$$\begin{aligned} c(a_n v) &= 2n - 2 & \qquad c(a_n b_n) &= 2n - 1 \\ c(a_{n-1} v) &= 2n - 3 & \qquad c(a_{n-1} b_{n-1}) &= 2n - 2. \end{aligned}$$

Therefore, c must induce an interval coloring of $F[\{a_i, b_i \colon i \in [n]\} \cup \{v\}] \simeq F_{n-2}$. However, for $n \equiv 1 \pmod 2$, F_{n-2} is not interval colorable by induction hypothesis, giving us a contradiction. ↯

Meanwhile, for $n \equiv 0 \pmod 2$, F_{n-2} is interval colorable. So, as there exists an interval coloring of $F[\{a_i, b_i \colon i \in [n]\} \cup \{v\}]$ such that the colors of edges incident to v form the interval $[2(n-2)]$ by Lemma 1, we conclude that c exists for $n \equiv 0 \pmod 2$.

This shows the first claim. For the second claim, we sketch the construction of a proper edge coloring of F_n with $\Delta(F_n)$ colors for $n \geq 2$: Color the edges incident to the center vertex v from 1 to $\Delta(F_n)$. As we have $\Delta(F_n) = 2n > 2$ colors, we can color each remaining edge using a color that is not used on adjacent edges. $\square$

1.2. Bounds on $w(G)$ and $W(G)$. In the context of schedulings, we have implied that having a smaller makespan is generally more desirable. So, it is natural to investigate $w(G)$ and $W(G)$.

Recall that $w(G)$ and $W(G)$ denote the smallest and largest $t \in \mathbb{N}$ for which G is t-interval colorable respectively (see Definition 6).

The following results have been established:

THEOREM 4 ([10]). *Let G be an interval colorable graph.*

- $W(G) \leq 2\,|V(G)| - 1.$
- $W(G) \leq (\operatorname{diam}(G) + 1)(\Delta(G) - 1) + 1.$
- *If G is triangle-free, then* $W(G) \leq |V(G)| - 1.$

- *If G is bipartite with parts A and B, then*
$$W(G) \leq \frac{1}{2}(\mathrm{diam}(G)+1)\Delta_A(G) + \frac{1}{2}(\mathrm{diam}(G)-1)\Delta_B(G) - \mathrm{diam}(G) + 1$$
if $\mathrm{diam}(G)$ *is odd, and if* $\mathrm{diam}(G)$ *is even*
$$W(G) \leq \frac{1}{2}\,\mathrm{diam}(G)(\Delta_A(G) + \Delta_B(G)) - \mathrm{diam}(G) + 1,$$
where $\Delta_A(G) = \max_{a\in A} \deg(a)$ *and* $\Delta_B(G) = \max_{b\in B} \deg(b)$.

- *In particular, if G is bipartite, then*
$$W(G) \leq \mathrm{diam}(G)\Delta(G) - \mathrm{diam}(G) + 1.$$

In a later paper, Giaro, Kubale and Malafiejski in [29] generalized the diameter bound for general graphs and were also able to improve the bound only depending on $|V(G)|$:

Theorem 5 ([29, Prop. 3.3 & Thm. 3.6]). *Let G be an interval colorable n-vertex graph. If $n \geq 3$, then $W(G) \leq 2n - 4$.*

More bounds can be found in [18, 42].

2. Results for certain graph classes

2.1. Forests and trees. One class of interval colorable graphs are forests, i.e. acyclic graphs. As trees are by definition the connected components of a forest, we may first look at trees.

Theorem 6. *Every tree T is $\Delta(T)$-interval colorable. In particular, $\theta_{\mathrm{int}}(T) = 1$.*

Proof. We proceed by induction on $m := |E(T)|$. Clearly, the statement holds for $m = 0$. Now, let $m > 0$ and assume that the statement holds for $m-1$. As T has at least two vertices, T must have a leaf $v \in V(T)$. Let $u \in V(T)$ be the unique vertex with $uv \in E(T)$ and let $T' := T - v$. By induction hypothesis, T' has a $\Delta(T')$-interval coloring c'. Also, since v was a leaf, $\Delta(T') \in \{\Delta(T)-1, \Delta(T)\}$.

Case 1: $\Delta(T') = \Delta(T) - 1$. Then u must be the only vertex in T with $\deg(u) = \Delta(T)$. In particular, we get $P_{c'}(u) = [\Delta(T)-1]$. So,
$$c\colon E(T) \to [\Delta(T)], e \mapsto \begin{cases} \Delta(T) & , e = uv \\ c'(e) & , \text{otherwise} \end{cases}$$
defines a $\Delta(T)$-interval coloring.

Case 2: $\Delta(T') = \Delta(T)$. As $P_{c'}(u)$ is an interval and proper subset of $[\Delta(T)]$, there exists a color $l \in [\Delta(T)] \setminus P_{c'}(u)$ such that $P_{c'}(u) \cup \{l\}$ is still an interval. So,
$$c\colon E(T) \to [\Delta(T)], e \mapsto \begin{cases} l & , e = uv \\ c'(e) & , \text{otherwise} \end{cases}$$
defines a $\Delta(T)$-interval coloring.

This shows the claim. □

Corollary 2. *Every forest F is $\Delta(F)$-interval colorable.*

Proof. Let $T_1, \dots, T_l$ be the connected components of F. As these are by definition trees, T_i has a $\Delta(T_i)$-interval coloring c_i for all $i \in [l]$ by Theorem 6. Thus,

$$c\colon E(F) \to [\Delta(F)], e \mapsto c_i(e) \text{ if } e \in E(T_i), i \in [l]$$

defines an $\Delta(F)$-interval coloring of F. Indeed, as $\Delta(F) = \max\{\Delta(T_i) \mid i \in [l]\}$ and each c_i is an $\Delta(T_i)$-interval coloring, c is an interval coloring with

$$c(E(F)) = \bigcup_{i=1}^{l} [\Delta(T_i)] = [\Delta(F)].$$

This concludes the proof. □

Note that this also shows that paths, caterpillars and stars are interval colorable. As the question of interval colorability and therefore also interval thickness is completely answered for trees and in general forests, people became interested in the parameters $w(T)$ and $W(T)$ for a tree T. This question was answered by Kamalian in [36]:

Theorem 7 ([36, Thm. 2]). *Let T be a tree and define*

$$M(T) \coloneqq \max\{|E(C)| \mid C \subseteq T, C \textit{ is a caterpillar.}\}.$$

Then we have

1. $w(T) = \Delta(T)$,
2. $W(T) = M(T)$,
3. $\operatorname{spec}(T) = [w(T), W(T)]$.

Proof idea. Clearly, $w(T)$ must be at least $\Delta(T)$ as each interval coloring is also a proper edge coloring. On the other hand, $w(T) = \Delta(T)$ due to Theorem 6. This shows 1.

For 2., one first shows that $W(T) \leq M(T)$: Let c be a $W(T)$-interval coloring of T and $e = xy, e' = x'y' \in E(T)$ be colored 1 and $W(T)$ respectively. W.l.o.g. we may assume $d(x, x') = \min\{d(u, v) \mid u \in \{x, y\}, v \in \{x', y'\}\}$. Consider the unique x-x'-path $P = v_0 \dots v_n$ with $v_0 = x, v_n = x'$. By minimality, $y, y' \notin V(P)$.

Applying Lemma 3 to yPy', we get

$$W(T) = c(x'y') \leq c(xy) + \sum_{k=0}^{n} (\deg(v_k) - 1) = \deg(v_0) + \sum_{k=1}^{n} (\deg(v_k) - 1).$$

But the right expression is just the size of the caterpillar induced by the edges

$$\{uv \in E(T) \mid uv \cap V(P) \neq \emptyset\}.$$

As $M(T)$ is the maximum number of edges a caterpillar in T can have, the inequality follows. To show equality for 2. and also 3., it suffices to show that for $t \in [\Delta(T), M(T)]$ there exists a t-interval coloring of T. This can be done by doing induction on $|E(T)|$ where in the induction step one may distinguish on whether T is itself a caterpillar or not. □

2.2. Regular graphs. As shown in the introduction, being Class 1 is in general a weaker property than being interval colorable. However, for regular graphs the two properties turn out to be equivalent:

THEOREM 8 ([10, Cor. 2]). *A regular graph is interval colorable if and only if it is Class 1.*

PROOF. By Theorem 3, every interval colorable, regular graph must be Class 1. On the other hand, if a graph G is regular and Class 1, there exists a $\Delta(G)$-edge coloring of G that by regularity is also a $\Delta(G)$-interval coloring, so G would be interval colorable. $\square$

From that theorem, we can conclude that cycles are interval colorable if and only if they are even. As C_n can be edge-decomposed into P_{n-1} and P_2 (ignoring isolated vertices), we get:

COROLLARY 3. *For $n \geq 3$, we have*

$$\theta_{\text{int}}(C_n) = \begin{cases} 1, & n \equiv 0 \pmod 2 \\ 2, & n \equiv 1 \pmod 2. \end{cases}$$

We also know that cliques are Class 1 if and only if the order is even, and that generally we can edge-decompose K_n into K_{n-1} and a star with $n-1$ leaves (ignoring isolated vertices). Thus, we get:

COROLLARY 4. *For $n \in \mathbb{N}$, we have*

$$\theta_{\text{int}}(K_n) = \begin{cases} 1, & n \equiv 0 \pmod 2 \vee n = 1 \\ 2, & n \equiv 1 \pmod 2. \end{cases}$$

For results on $w(K_{2n})$, $W(K_{2n})$ and $\operatorname{spec}(K_{2n})$, we refer to [40, 52, 44].

What makes regular, interval colorable graphs somewhat special is that their spectrum is always an interval itself.

LEMMA 5 ([9, Prop. 2]). *Let G be a regular, interval colorable graph. Then*

$$\operatorname{spec}(G) = [w(G), W(G)] = [\Delta(G), W(G)].$$

PROOF. Nothing needs to be shown for $\Delta(G) \leq 1$, so assume $\Delta(G) > 1$. It suffices to show that if G is $(t+1)$-interval colorable for $w(G) \leq t < W(G)$, then G is t-interval colorable: Let c be a $(t+1)$-interval coloring of G. Note that if $t+1 \in P_c(v)$, then $t+1-\Delta(G) \notin P_c(v)$ and $t+1-(\Delta(G)-1) \in P_c(v)$ for every $v \in V(G)$. Thus,

$$c' \colon E(G) \to [t], e \mapsto \begin{cases} t+1-\Delta(G), & c(e) = t+1 \\ c(e), & \text{otherwise} \end{cases}$$

is a proper edge-coloring of G. In particular, every palette of c' forms an interval. $\square$

However, regularity doesn't seem to be a property that makes the problem of determining the interval thickness exactly or bounding it any easier. Apart from determining whether a regular graph is Class 1 or not already being $\mathcal{NP}$-hard (see [48]), we note the following.

LEMMA 6. *Let G be a regular graph that is Class 2 and let $G_1, \ldots, G_k$ be an edge-decomposition of G into interval colorable graphs. Then at least two of the graphs in the decomposition are not regular.*

PROOF. First, assume that G_i is regular for all $i \in [n]$. By Theorem 8, we know that there exists an edge coloring $c_i \colon E(G_i) \to [\Delta(G_i)]$ for every $i \in [k]$. But then

$$c\colon E(G) \to [\Delta(G)], e \mapsto \begin{cases} c_1(e), & e \in E(G_1) \\ c_j(e) + \sum_{i=1}^{j-1} \Delta(G_i), & e \in E(G_j), j > 1. \end{cases}$$

defines an edge coloring of G using $\Delta(G) = \Delta(G_1) + \cdots + \Delta(G_k)$ colors. ↯

So, there exists $j \in [k]$ such that G_j is not regular. But that means

$$G - E(G_j) = \bigcup_{i \in [k], i \neq j} G_i$$

is not regular as well, meaning that there must be at least another $i \in [k] \setminus \{j\}$ for which G_i is not interval colorable. □

Another approach one may think of for the Class 2 case is to exploit the fact that for a Δ-regular, Class 2 graph and a $(\Delta + 1)$-edge coloring of the graph each vertex "misses" exactly one color:

Let us first consider $\Delta \geq 3$ and a Δ-regular graph G with $(\Delta + 1)$-edge coloring c of G where c only "induces" 3 different kinds of palettes, i.e. $|\{P_c(v) \colon v \in V(G)\}| = 3$. As every vertex "misses" exactly one color, we can permute the colors such that $\{P_c(v) \colon v \in V(G)\} = \{[\Delta + 1] \setminus i \colon i \in \{1, 2, \Delta + 1\}\}$, i.e. the only colors missing at some vertex are $1, 2$ and $\Delta + 1$. It is then easy to see that the graph induced by color classes 1 and 2 and the graph induced by color classes 3 to $\Delta + 1$ edge-decompose G and are both interval colorable (see Figure 2).

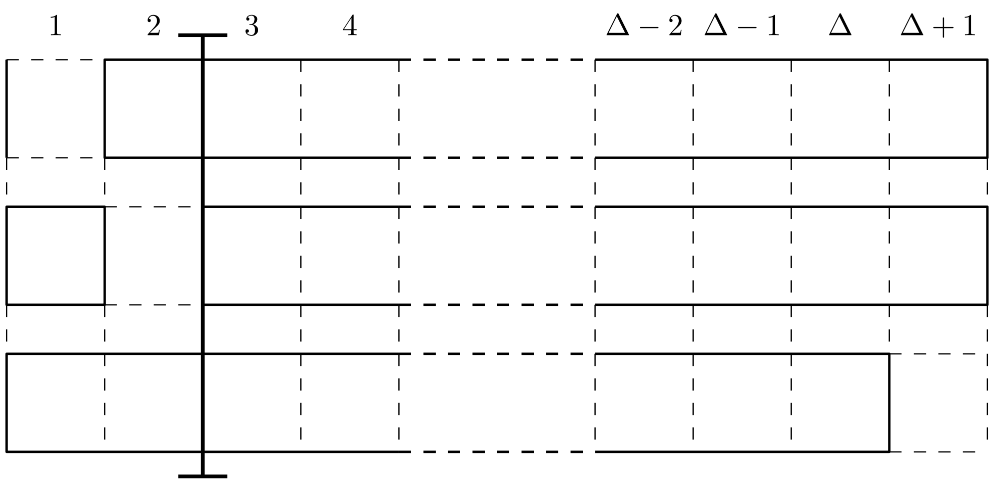

FIGURE 2. The three palettes of G and the "split" between 2 and 3

We can generalize this approach to an arbitrary number of different palettes induced by the coloring:

Lemma 7. *Let G be an r-regular, Class 2 graph and let $c\colon E(G) \to [r+1]$ be an edge coloring of G with k distinct palettes, i.e. $|P_c(v)\colon v \in V(G)| = k$. Then,*

$$\theta_{\mathrm{int}}(G) \leq \left\lceil \frac{k}{2} \right\rceil .$$

Proof. Note that for every $v \in V(G)$ there is exactly one color $i(v) \in [r+1]$ not in $P_c(v)$. So, by suitably permuting the colors, we may assume that

$$\{i(v)\colon v \in V(G)\} = [k-1] \cup \{r+1\} .$$

For $1 \leq j \leq \lceil k/2 \rceil$, let $G_j \coloneqq (V(G), E(G_j))$ where

$$E(G_j) \coloneqq \begin{cases} c^{-1}(\{2j-1, 2j\}), & j < \lceil k/2 \rceil \\ c^{-1}(\{2\lceil k/2 \rceil - 1, \ldots, r+1\}), & j = \lceil k/2 \rceil . \end{cases}$$

Let $c_j \coloneqq c\big|_{E(G_j)}$ for $[\lceil k/2 \rceil]$. We will show that c_j is an interval coloring of G_j:

For $j < \lceil k/2 \rceil$, $c_j(v)$ is a subset of $\{2j-1, 2j\}$, so it's in particular an interval.

For $j = \lceil k/2 \rceil$, we have

$$P_{c_j}(v) = \begin{cases} [2\lceil k/2 \rceil - 1, r+1], & i(v) < k-1 \\ [k, r+1], & i(v) = k-1 \\ [2\lceil k/2 \rceil - 1, r], & i(v) = r+1. \end{cases}$$

Thus, all G_j are interval colorable, proving the claimed bound. □

So, as long as the number of distinct palettes is small, one may hope to get better bounds on $\theta_{\mathrm{int}}(G)$ for regular graphs G that are not Class 1.

However, it was shown that the number of distinct palettes of an edge coloring can be as big as possible if the degree of the graph is odd. More formally:

Theorem 9 ([50]). *For every $k \in \mathbb{N}$, there exists a $(2k+1)$-regular graph G such that for every edge coloring c of G we have*

$$|\{P_c(v)\colon v \in V(G)\}| \geq 2k+2.$$

2.3. Bipartite graphs. The case with bipartite graphs is of special interest in the study of interval colorings: In most applications, we can identify two types of entities with no edges between the same type. One may for example think of a school setting, where we want to schedule lectures such that they are consecutive for both teachers and classes (see Ex. 1 in [7]). For a broader view, we refer to chapter 3.

By Theorem 2, all bipartite graphs are Class 1. However, as with Class 1 graphs (see Lemma 4), it is also not the case that all bipartite graphs are interval colorable. Indeed, according to Petrosyan and Khachatrian in [56], the first published counterexample was by Sevastianov in [62]. As the original paper omits this, we will prove the correctness of Sevastianov's counterexample here.

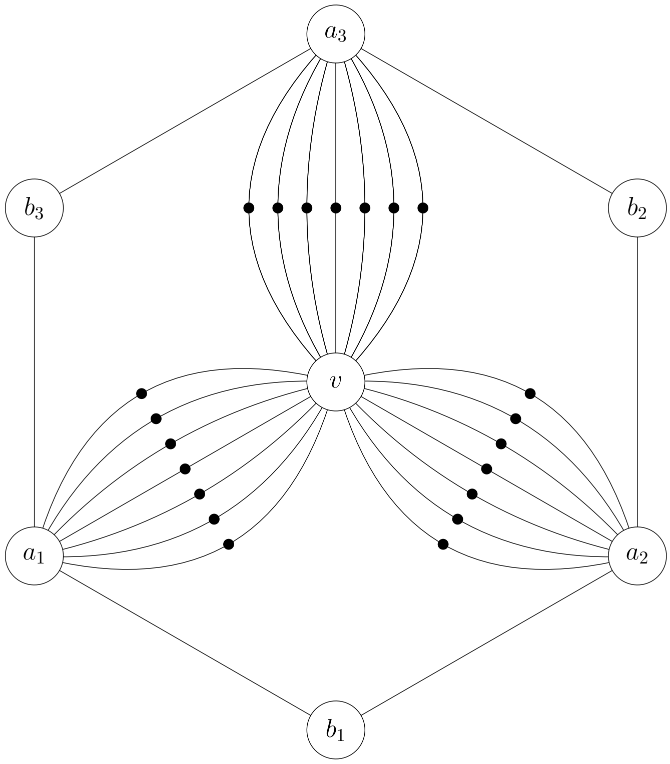

FIGURE 3. The bipartite Sevastianov graph

DEFINITION 11 (Sevastianov graph). *The Sevastianov graph S is defined by*

$$V(S) \coloneqq \{v\} \cup \{a_i, b_i \mid i \in [3]\} \cup \{u_{i,j} \mid i \in [3], j \in [7]\}$$
$$E(S) \coloneqq \{a_i b_i, a_{i+1} b_i \mid i \in [3]\} \cup \{v u_{i,j}, u_{i,j} a_i \mid i \in [3], j \in [7]\},$$

where the indices i are operated modulo 3.

REMARK 3. *As we can see in Figure 3, S is indeed bipartite with partite sets $\{v\} \cup \{a_i \mid i \in [3]\}$ and $\{b_i \mid i \in [3]\} \cup \{u_{i,j} \mid i \in [3], j \in [7]\}$. Furthermore, note that*

$$\Delta(S) = \deg(v) = 21, \deg(a_i) = 9, \delta(S) = \deg(b_i) = \deg(u_{i,j}) = 2$$

for all $i \in [3], j \in [7]$. Also note that $|V(S)| = 28$ and $|E(S)| = 50$.

THEOREM 10. *The Sevastianov graph S is not interval colorable.*

PROOF. For the sake of contradiction, let's assume that S was interval colorable. By Corollary 1 and S being connected, there exists some $t \geq \Delta(S) = 21$ for which S has a t-interval coloring c. Let $p \coloneqq \min P_c(v)$, i.e. $\max P_c(v) = p + \deg(v) - 1 = p + 20$. There exists $i, i' \in [3], j, j' \in [7]$ such that $c(u_{i,j} v) = p$ and $c(u_{i',j'} v) = p + 20$.

Note that, by Lemma 3, we have

$$P_c(u_{i',j'}) \subseteq [p + 20 - (\deg(u_{i',j'}) - 1), p + 20 + (\deg(u_{i',j'}) - 1)] = [p + 19, p + 21].$$

We now do the following case distinction.

Case 1: $i = i'$. Then $P = vu_{i,j}a_iu_{i,j'}$ is a v-$u_{i,j'}$-path. Using Lemma 3, we get

$$\begin{aligned} c(a_iu_{i,j'}) &\leq c(vu_{i,j}) + \deg(u_{i,j}) + \deg(a_i) - 2 \\ &= p + 9 \notin [p + 19, p + 21]. \lightning \end{aligned}$$

Case 2: $i \neq i'$. Then $P = vu_{i,j}a_ib_ka_{i'}u_{i',j'}$ is a v-$u_{i,j'}$-path for some $k \in [3]$.
We proceed similarly to Case 1: Lemma 3 implies

$$\begin{aligned} c(a_{i'}u_{i',j'}) &\leq c(vu_{i,j}) + \sum_{x \in \{u_{i,j}, a_i, b_k, a_{i'}\}} (\deg(x) - 1) \\ &= p + 1 + 8 + 1 + 8 \\ &= p + 18 \notin [p + 19, p + 21]. \lightning \end{aligned}$$

Thus, S is not interval colorable. □

Corollary 5. $\theta_{\mathrm{int}}(S) = 2$.

Proof. As S is not interval colorable, $\theta_{\mathrm{int}}(S) > 1$. On the other hand, let E' be the set of edges incident to v and $E'' := E(S) \setminus E'$. Define

$$G' := (V(S), E') \qquad\qquad G'' := (V(S), E'').$$

G' is a star, in particular a tree, so it's interval colorable. Furthermore, G'' is a C_6, which is interval colorable, with additional pendant edges which do not change the interval colorability. So, S edge-decomposes into two interval colorable graphs. □

Generalizations of Sevastianov's counterexample can be found in [15, 30].

Other counterexamples were given by Erdős, Hertz and de Werra, and Malafiejski, all of them being generalized to obtain a respective family of non-interval colorable, connected, bipartite graphs in [56], for Hertz's in particular in [14, 30]. The first two counterexamples are also described in [35] in the section 12.23 "Scheduling without waiting periods". Various other constructions were also proposed in [56]. In particular, the following was shown:

Theorem 11 ([56]). *For any integer $\Delta \geq 11$, there is a connected bipartite graph G that is not interval colorable with $\Delta(G) = \Delta$.*

Similar questions were also answered for bipartite multigraphs in that same paper.

However, there are also classes of bipartite graphs known to be interval colorable. It is known, due to computer search performed by Khachatrian and Mamikonyan in [43] and Giaro in [26], that every bipartite graph of order at most 15 is interval colorable. In terms of the maximum degree, Hansen proved the following:

Theorem 12 ([31]). *All subcubic, bipartite graphs are interval colorable with at most 4 colors.*

The proof of this can also be found in [47, pp. 209–210] and was independently reproven in [30]. Furthermore, we obtain an algorithm that finds a desired coloring in $\mathcal{O}(|E|\sqrt{|V|})$ time from the proof.

More generally, every regular bipartite graph is interval colorable by Theorem 2 and 8 such as the hypercube Q_n for which it was shown:

Theorem 13 ([52, 57]). *For every* $n \in \mathbb{N}$, $\operatorname{spec}(Q_n) = [n, n(n+1)/2]$.

In a natural progression, people then became interested in the biregular case.

Definition 12 (Biregularity, (a,b)-biregularity). *We say that a graph G is biregular if G is bipartite with partite sets A and B such that all vertices in each partite set have the same degree. In particular, if for all $u \in A, v \in B$ we have $\deg(u) = a$ and $\deg(v) = b$, then we call G (a,b)-biregular.*

Conjecture 1 ([35]). *Every (a,b)-biregular graph is interval colorable.*

Some graphs that are biregular and interval colorable are complete bipartite graphs.

Theorem 14 ([10, Lem. 1]). $\forall\, m, n \in \mathbb{N}$: $\theta_{\text{int}}(K_{m,n}) = 1$.

Proof. Let the parts of $K_{m,n}$ be $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$. Define
$$c\colon E(K_{m,n}) \to [m+n-1], a_i b_j \mapsto i + j - 1.$$
It is clear that $P_c(a_i) = [i, i+n-1]$ and $P_c(b_j) = [j, j+m-1]$ for all $i \in [m], j \in [n]$. Since $\deg(a_i) = n$ and $\deg(b_j) = m$ for all $i \in [m], j \in [n]$, c defines an $(m+n-1)$-interval coloring. □

For complete bipartite graphs, more is even known.

Theorem 15 ([36, Thm. 1]). $\forall\, m, n \in \mathbb{N}$, $w(K_{m,n}) = m+n-\gcd(m,n)$[1], $W(K_{m,n}) = m+n-1$ *and* $\operatorname{spec}(K_{m,n}) = [w(K_{m,n}), W(K_{m,n})]$.

However, apart from such special classes, the question remains generally unknown. Since Theorem 12 covers the (a,b)-biregular graphs for $a, b \in [3]$, the next case would be whether all $(3,4)$-biregular graphs are interval colorable. From Kőnig's Edge Coloring Theorem / Theorem 2 and Theorem 12, it is clear that all $(3,4)$-biregular graphs have interval thickness 2 as we can take a 4-edge coloring of our graph, take the subcubic, bipartite subgraphs induced by the color classes 1 to 3 and color class 4 respectively, which then by Theorem 12 are interval colorable. Nevertheless, it is still an open problem whether all $(3,4)$-biregular graphs are interval colorable.

We give three sufficient conditions for when a $(3,4)$-biregular graph is interval colorable:

Theorem 16 ([60]). *Let $G = (X \cup Y, E)$ be a $(3,4)$-biregular graph having a cubic subgraph covering the set Y, then G has an 6-interval coloring.*

However, deciding whether such a subgraph exists turns out to be $\mathcal{NP}$-complete which was also proven in [60].

[1]For $m, n \in \mathbb{N}$, $\gcd(m,n)$ denotes the *greatest common divisor* of m and n.

Theorem 17 ([11]). *Let $G = (X \cup Y, E)$ be a $(3,4)$-biregular graph having a spanning path factor whose components are paths with endpoints in X and lengths in $\{2,4,6,8\}$, then G has a 6-interval coloring.*

Theorem 18 ([64]). *Let $G = (X \cup Y, E)$ be a $(3,4)$-biregular graph having two $(2,3)$-biregular subgraphs $G_1 = (Y \cup X_1, E_1)$ and $G_2 = (Y \cup X_2, E_2)$ such that $X_1 \uplus X_2 = X, E_1 \uplus E_2 = E$. Then G has an 6-interval coloring.*

It is worth nothing that the authors in [11] conjecture that every $(3,4)$-biregular graph has such a spanning path factor as described in Theorem 17 with Casselgren even providing evidence to the conjecture by proving the following:

Theorem 19. *Every $(3,4)$-biregular graph has a spanning subgraph whose components are non-trivial paths with degree 3 vertices as endpoints and length at most 22.*

While the $(3,4)$-case is not completely solved, there have been various positive results for other biregular graphs.

Theorem 20 ([33, 31, 20, 20]).

- *All $(2,b)$-biregular graphs are interval colorable, $b \in \mathbb{N}$.*
- *All $(3,6)$-biregular graphs are 7-interval colorable.*
- *All $(3,9)$-biregular graphs having a cubic subgraph covering all vertices of degree 9 are 10-interval colorable.*

As the authors note in [20], the latter two results presented above are constructive and let us obtain the desired colorings in polynomial time.

Sufficient conditions for when a $(3,5)$-biregular graph is interval colorable can also be found in [19].

Concerning bounds on $w(G)$, Hanson and Loten were able to show that the lower bound established in Theorem 15 holds for general biregular graphs:

Theorem 21 ([32]). *If G is an (a,b)-biregular, interval colorable graph G, then*

$$w(G) \geq a + b - \gcd(a,b).$$

Furthermore, the upper bound for bipartite graphs given in Theorem 4 can be improved for the biregular case:

Theorem 22 ([5]). *If G is a connected, (a,b)-biregular, interval colorable graph G with $|G| \geq 2(a+b)$, then*

$$W(G) \leq |G| - 3.$$

The authors in [5] also show that the bounds given in Theorem 21 and 22 are tight by constructing for every $n \in \mathbb{N}$ and $a > b \geq 3$ a connected (a,b)-biregular graph G with $n(a+b)$ vertices such that $[a+b-\gcd(a,b), n(a+b)-3] \subseteq \operatorname{spec}(G)$.

Finally, it has been shown that doubly convex bipartite graphs (see [8, 37]) and outerplanar bipartite graphs (see [28]) are interval colorable.

2.4. Other classes. We will conclude this section by presenting results for lesser studied graph classes:

2.4.1. *Planar graphs.* As mentioned, it has been shown that bipartite outerplanar graphs are interval colorable (see [28]). In a more general scheme, Axenovich considered in [12] planar interval colorable graphs, for which she obtained:

THEOREM 23 ([12, Thm. 16]). *For every planar interval colorable graph G of order n, $W(G) \leq (11/6)n$.*

Axenovich also showed that every outerplanar graph of order at least 4 without a separating triangle[2] is interval colorable. It was also in [12] conjectured that for all outerplanar triangulations of order at least 4 having no separating triangle is also a necessary condition. The answer turns out to be negative, as Petrosyan in [54] constructs a class of interval colorable outerplanar triangulations of order at least 4 with separating triangles.

In [54], Petrosyan also showed that if G is a 2-connected outerplanar graph G with $\Delta(G) = 3$, then

$$w(G) = \begin{cases} 3, & |V(G)| \equiv 0 \pmod 2 \\ 4, & |V(G)| \equiv 1 \pmod 2. \end{cases}$$

We note that by Theorem 30 and the fact that every such graph is Class 1 (as stated in [7]), they are indeed interval colorable.

2.4.2. *Complete k-partite graphs.* In a natural progression from complete graphs (see Corollary 4), Kamalian and Petrosyan considered in [41] the k-partite K_{n*k} where each part has n vertices. In particular, using Theorem 8, it was determined that K_{n*k} is interval colorable if and only if nk is even. They also analyzed the spectrum of K_{n*k} for even nk and showed

$$\operatorname{spec}(K_{n*k}) \supseteq \begin{cases} [(k-1)n, (3k/2-1)n-1], & k \equiv 0 \pmod 2 \\ [(k-1)n, (2k-p-q)n-1], & k = p \cdot 2^q \text{ where } p \text{ is odd and } q \in \mathbb{N}. \end{cases}$$

2.4.3. *Grids, Cylinders, Tori and more.* Remarkably, interval colorability is *closed* under the operation of taking *Cartesian products*:

DEFINITION 13 (Cartesian product of graphs). *For graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$, we define $G \square H$ by*

$$\begin{aligned} V(G \square H) &\coloneqq V_1 \times V_2 \\ E(G \square H) &\coloneqq \{\{(v,u),(v,w)\} : v \in V_1, uw \in E_2\} \\ &\qquad \cup \{\{(v,u),(w,u)\} : u \in V_2, vw \in E_1\}. \end{aligned}$$

THEOREM 24 ([28, Thm. 2.4]). *If $G = (V_1, E_1)$ and $H = (V_2, E_2)$ are are interval colorable, then $G \square H$ is also interval colorable. Moreover, if G and H have an interval coloring using r_1 and r_2, then $G \square H$ has an interval coloring using at most $r_1 + r_2$ colors.*

[2] A separating triangle is defined as a triangular face none of whose edges belong to the unbounded face of the planar embedding.

PROOF. Let c_1 and c_2 be interval colorings of G and H using r_1 and r_2 colors respectively. We define

$$c\colon E(G \square H) \to \mathbb{Z}, e \mapsto \begin{cases} c_1(vw) + \min P_{c_2}(u), & e = \{(v,u),(w,u)\} \\ c_2(uw) + \max P_{c_1}(v) + 1, & e = \{(v,u),(v,w)\}. \end{cases}$$

Now, fix arbitrary $(v,u) \in V(G \square H)$. Considering the neighboring vertices with second component u, the set of colors of the corresponding edges is

$$P_{c_1}(v) + \min P_{c_2}(u) = [\min P_{c_1}(v) + \min P_{c_2}(u), \max P_{c_1}(v) + \min P_{c_2}(u)].$$

Considering the neighboring vertices with first component v, the set of colors of the corresponding edges is

$$P_{c_2}(u) + \max P_{c_1}(v) + 1 = [\min P_{c_2}(u) + \max P_{c_1}(v) + 1, \max P_{c_2}(u) + \max P_{c_1}(v) + 1].$$

Thus, in total, $P_c((v,u)) = [\min P_{c_1}(v) + \min P_{c_2}(u), \max P_{c_1}(v) + \max P_{c_2}(u) + 1]$, which in particular contains $\deg((v,u)) = \deg_G(v) + \deg_H(u)$ colors.

Hence, c is an interval coloring of $G \square H$ with

$$c(E(G \square H)) \subseteq [\min c_1(E_1) + \min c_2(E_2), \max c_1(E_1) + \max c_2(E_2) + 1].$$

In particular, it uses at most $r_1 + r_2$ colors. $\square$

Note that this implies that $w(G \square H) \leq w(G) + w(H)$ for interval colorable G, H. This result can be made even more precise:

THEOREM 25 ([45]). *If G, H are interval colorable, then $w(G \square H) \leq w(G) + w(H)$ and $W(G \square H) \geq W(G) + W(H)$.*

Grids $G(n_1, \ldots, n_k) := P_{n_1} \square \ldots \square P_{n_k}, n_i \in \mathbb{N}$, *cylinders* $C(n_1, n_2) := P_{n_1} \square C_{n_2}$ and *tori* $T(n_1, n_2) := C_{n_1} \square C_{n_2}$ were then investigated in [27], where they showed:

THEOREM 26 ([27]). *Let $G = G(n_1, \ldots, n_k)$ or $G = C(m, 2n), m \in \mathbb{N}, n \geq 2$, or $T(2m, 2n), m, n \geq 2$, then G is interval colorable and $w(G) = \Delta(G)$.*

For lower bounds on the maximum span of grids, cylinders and toris, we refer to [57]. It is also interesting to note that, using the general results established in [57], they were able to prove that

$$W(Q_n) = W(\underbrace{K_2 \square \ldots \square K_2}_{n \text{ times}}) = \frac{n(n+1)}{2}.$$

A more general result can be found in [53]:

THEOREM 27 ([53]). *If G is interval colorable and H is r-regular, then $G \square H$ is interval colorable. Moreover, $w(G \square H) \leq w(G)r$ and $W(G \square H) \geq W(G)r$.*

The interval colorability of other types of graph products and other Cartesian products were also investigated in [53, 59].

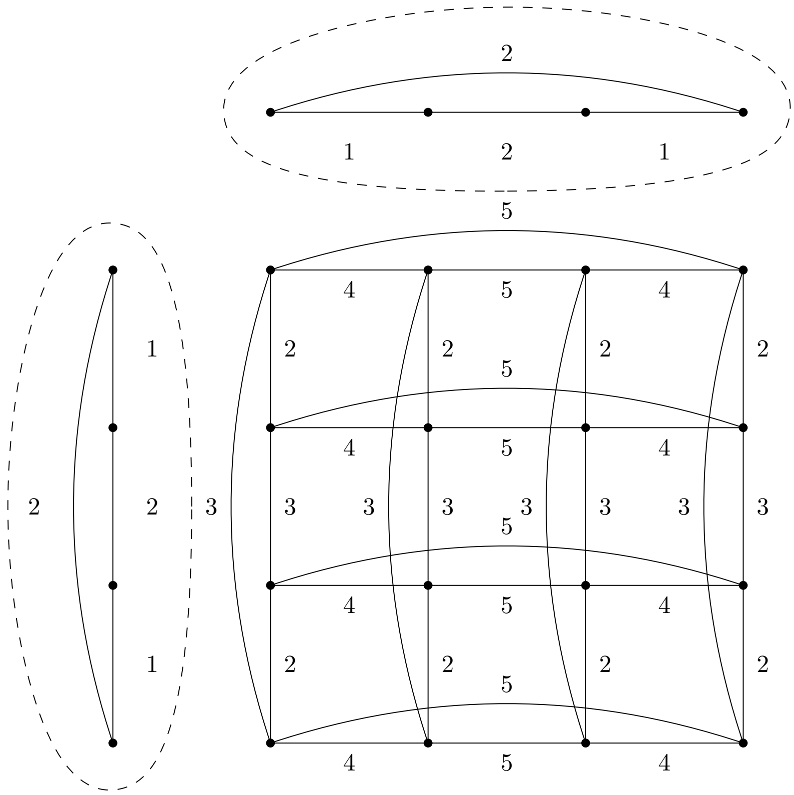

Figure 4. Coloring of $T(4,4)$ by Construction in Theorem 24

3. Complexity theoretic results

While some classes of graphs are interval colorable where we can even construct the coloring in polynomial time (some of them given in [28]), finding an interval coloring or deciding whether a graph has an interval coloring is in general a very hard problem. We note some of the most important complexity theoretic results here.

3.1. The decision problem of interval colorability. It is well-known that the decision problem to determine whether a graph is Class 1 or not is $\mathcal{NP}$-complete even for 3-regular graphs, as shown by Holyer in [34]. Building up on Holyer's paper, Leven and Galil in [48] then showed that the decision problem for k-regular graphs is in fact $\mathcal{NP}$-complete for any $k \geq 3$. As a result, by Theorem 8, the decision problem to determine whether a k-regular graph is interval colorable or not is also $\mathcal{NP}$-complete for any $k \geq 3$. Unlike Class 1 graphs however, Sevastianov showed in [62] that the question is even $\mathcal{NP}$-complete for bipartite graphs. For that, Sevastianov showed that the strongly $\mathcal{NP}$-complete problem Sequencing Within Intervals [24, p. 70] can be (pseudo-polynomially) reduced to deciding whether a bipartite graph is interval colorable. Informally speaking, the problem asks one to decide whether to a given set of tasks with release times and deadlines there exists a valid schedule on a single processor. For more details, we refer to our translation.

3.2. The decision problem of Δ-interval colorability. As our main goal, apart from finding an interval coloring, is for the number of colors used to be as small as possible, the problem of finding $w(G)$ was studied.

Recall that $w(G) \geq \Delta(G)$ since interval colorings are in particular proper.

As it turns out, for a lot of different graph classes, determining $w(G)$ is $\mathcal{NP}$-hard as determining for those classes whether a graph G is $\Delta(G)$-interval colorable is already $\mathcal{NP}$-complete.

More quantitatively, Giaro showed in [25] the following:

THEOREM 28 ([25]). *The problem of deciding the existence of a $\Delta(G)$-interval coloring of a bipartite graph G can be solved in polynomial time if $\Delta(G) \leq 4$ and is $\mathcal{NP}$-complete if $\Delta(G) \geq 5$.*

A proof of this can also be found in [47, ch. 9.2].

Even more surprisingly, it turns out that even if we only consider graph classes that are known to be interval colorable, i.e. where $w(G)$ is always defined, it can be $\mathcal{NP}$-hard to determine $w(G)$:

THEOREM 29 ([5, 20, 20]).

- *The problem of deciding whether a given $(3, 6)$-biregular graph is 6-interval colorable is $\mathcal{NP}$-complete.*
- *The problem of deciding whether a given $(3, 9)$-biregular graph is 9-interval colorable is $\mathcal{NP}$-complete.*
- *The problem of deciding whether a given $(4, 8)$-biregular graph is 8-interval colorable is $\mathcal{NP}$-complete.*

This kind of behaviour resembles that of determining whether a graph is Class 1 or Class 2: On the one hand, Theorem 20 implies that every $(3, 6)$-biregular graph G is $(\Delta(G) + 1)$-interval colorable and that the coloring can be obtained in polynomial time. On the other hand, it is $\mathcal{NP}$-complete to determine whether such a graph is $\Delta(G)$-interval colorable.

We also note that even though there is no algorithm that can decide those problems in polynomial time (under the assumption $\mathcal{P} \neq \mathcal{NP}$), it is somewhat easy to characterize them: Consider for example the last result. On the one hand, every 8-interval coloring of a $(4, 8)$-biregular graph $G = (A \cup B, E)$ induces a 4-regular subgraph covering B using the edges colored $\{1, 2, 3, 4\}$. On the other hand, every $(4, 8)$-biregular graph $G = (A \cup B, E)$ with a 4-regular subgraph H covering B is 8-interval colorable: H and $G - E(H)$ are by Theorem 8 4-interval colorable and thus naturally induce an 8-interval coloring.

Generally, the following holds:

LEMMA 8 ([5, 20, 20]).

- *A $(3, 6)$-biregular bipartite graph G has an 6-interval coloring if and only if G has a cubic subgraph covering all vertices of degree 6 in G.*
- *A $(3, 9)$-biregular graph G has an interval 9-coloring if and only if it admits a decomposition into three edge-disjoint 3-regular subgraphs.*

- *A* $(4,8)$*-biregular graph* G *has an interval 8-coloring if and only if it has a 4-regular subgraph covering the vertices of degree 8.*

4. Upper bounds on the interval thickness

Clearly, as determining whether a given graph is interval colorable or not is generally hard, the same goes for determining the interval thickness. That's why upper bounds on it are of particular interest.

Recall that the interval thickness of a graph is defined as the minimum number k for which the graph is edge-decomposable into k interval colorable subgraphs (see Definition 8).

The usual approach for that is to use some particular classes of graphs which are known to be interval colorable and to then edge-decompose our graph into subgraphs belonging to one of those classes:

LEMMA 9. *For any* n*-vertex graph* G *and subgraphs* $G', G'' \subseteq G$ *with* $E(G') \,\dot\cup\, E(G'') = E(G)$*. Then*

$$\theta_{\text{int}}(G) \leq \theta_{\text{int}}(G') + \theta_{\text{int}}(G'').$$

PROOF. As adding isolated vertices to a graph doesn't change the interval thickness, we may assume that $V(G') = V(G'') = V(G)$. Let $k = \theta_{\text{int}}(G')$ and $l = \theta_{\text{int}}(G'')$. By definition, there exists edge-decompositions $G'_1, \ldots, G'_k \subseteq G'$, $G''_1, \ldots, G''_l \subseteq G''$ of G' and G'' respectively such that all G'_i's and G''_j's are interval colorable. As G' and G'' edge-decompose G, $G'_1, \ldots, G'_k, G''_1, \ldots, G''_l$ is also an edge-decomposition of G with each graph being interval colorable. Thus, $\theta_{\text{int}}(G) \leq k + l$ as claimed. $\square$

From that, one can get a fairly simple upper bound from Vizing's Theorem 1: If a graph G has maximum degree Δ, then we can edge-decompose it into $\Delta+1$ matchings corresponding to the color classes of a $(\Delta+1)$-edge coloring of G, each of which is interval colorable. So, $\theta_{\text{int}}(G) \leq \Delta + 1$.

Of course, this bound is very unsatisfactory: If a graph is very dense, then the maximum degree is roughly equal to the order of the graph. Applying this to our motivating Example 1, it would imply that the number of days needed to distribute the talks is roughly the number of people attending the meeting.

A better bound is presented in [7] where the notion was first introduced.

For that, a new class of interval colorable graphs is established:

THEOREM 30 ([7, Thm. 2.1]). *Let* G *be a graph with* $\chi'(G) = \Delta(G) \leq 3$*. If no component of* G *is an odd cycle, then* G *is interval colorable.*

This result is on its own of great significance.

- It generalizes Theorem 12 by Hansen: All bipartite graphs are by Kőnig's Edge Coloring Theorem / Theorem 2 Class 1 and famously don't contain any odd cycles[3].

[3]See Proposition 1.6.1. in [22].

- Note that this allows us to classify interval colorable graphs with maximum degree at most 3: If a graph is Class 2, we know by Theorem 3 that they are not interval colorable. If our graph is Class 1, then the condition given in the Theorem is clearly necessary, but also shown to be sufficient.

- If we restrict ourselves further to connected graphs, then it shows that in this case being Class 1 and being interval colorable are also equivalent. It turns out that this result is "tight" with respect to the maximum degree:

LEMMA 10. *There exists a connected, Class 1 graph of maximum degree 4 that is not interval colorable.*

PROOF. Consider the graph G consisting of a triangle $T = v_1v_2v_3v_1$ and between any two vertices v_i, v_j, $1 \leq i < j \leq 3$, a path of length 2 with internal vertex $v_{i,j}$.

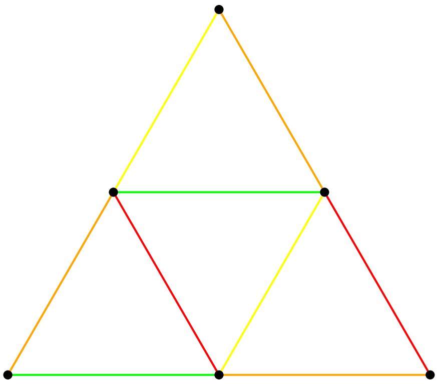

FIGURE 5. Edge coloring of G

Clearly, $\Delta(G) = 4$ and from Figure 5 we can see that G is Class 1. It remains to show that G is not interval colorable: For the sake of contradiction, assume that there exists an interval coloring φ of G. For φ being an interval coloring,

(1) the parity of the two colors in $P_\varphi(v_{i,j})$ must be different for all $1 \leq i < j \leq 3$,

(2) two colors in $P_\varphi(v_k)$ are even and two are odd for $k \in [3]$.

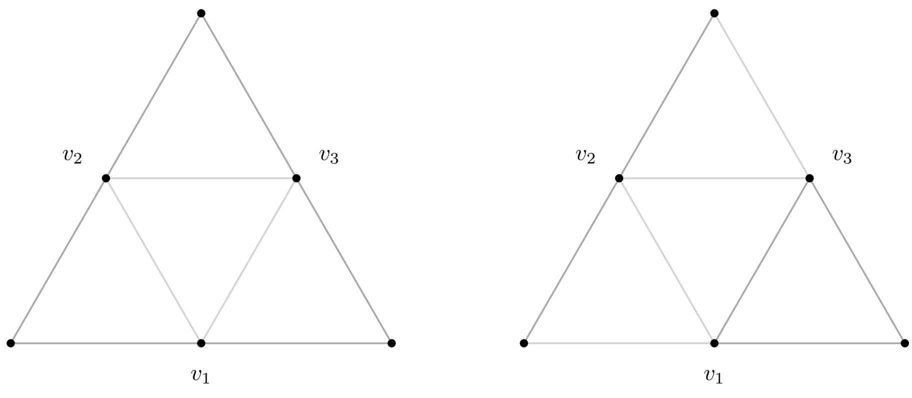

FIGURE 6. Case 1 and Case 2, where light gray means the color is even and dark gray means that it's odd

Case 1: $\varphi(v_1v_2), \varphi(v_2v_3), \varphi(v_3v_1)$ all have the same parity, w.l.o.g. they are all even. Then, by (2), we must have that $\varphi(v_1v_{1,2}), \varphi(v_2v_{1,2})$ both are odd, contradicting (1) at $v_{1,2}$. ↯

Case 2: Exactly two of the edges in $E(T)$ are of the same parity. Up to symmetry, we may assume that $\varphi(v_1v_2), \varphi(v_2v_3)$ are even while $\varphi(v_3v_1)$ is odd. Applying (2) for v_2, this implies that $\varphi(v_2v_{1,2})$ and $\varphi(v_2v_{2,3})$ are both odd. For $v_{1,2}$ and $v_{2,3}$, $\varphi(v_1v_{1,2})$ and $\varphi(v_3v_{2,3})$ must be even by (1). As a result, since (2) holds for v_1 and v_3, we get that $\varphi(v_1v_{1,3})$ and $\varphi(v_3v_{1,3})$ must both be odd in return. This contradicts (1) for $v_{1,3}$. ↯

Thus, G is not interval colorable. □

Using Theorem 30, by decomposing graphs into subgraphs known to be interval colorable, they were then able to show:

THEOREM 31 ([7, Thm. 3.3]). *For any graph $G, \theta_{\text{int}}(G) \leq 2\lceil \chi'(G)/5 \rceil$.*

Since by Vizing's Theorem / Theorem 1, $\chi'(G) \in \{\Delta(G), \Delta(G)+1\}$, this is an improvement in terms of the factor in front of $\Delta(G)$.

It is also good to note that this general bound is somewhat constructive: Vizing's Theorem / Theorem 1 can be turned into an algorithm that lets us compute a $(\Delta + 1)$-edge coloring of a graph in polynomial time and the proofs of Theorem 30 and therefore also Theorem 31 can be turned into polynomial algorithms in the graph and its $(\Delta+1)$-edge coloring, giving us:

COROLLARY 6. *There exists a polynomial-time algorithm that that computes for any graph G an edge decomposition into $2\lceil(\Delta(G)+1)/5\rceil$ subgraphs that are all interval colorable.*

For a bipartite graph G, since G does not contain any odd cycles and by Kőnig's Edge Coloring Theorem / Theorem 2 has a $\Delta(G)$-edge coloring, by grouping 3 (or possible less) color classes we get $\lceil \Delta(G)/3 \rceil$ subcubic Class 1 graphs. Thus, Theorem 30 gives us $\theta_{\text{int}}(G) \leq \lceil \Delta(G)/3 \rceil$. For practical purposes, one may want the several schedules to be of roughly equal "size".

THEOREM 32 ([7, Prop. 3.5]). *If G is a bipartite graph with $\Delta(G) \geq 4$, then $\theta_{\text{int}}(G) \leq \lceil \Delta(G)/3 \rceil$. Moreover, G can be edge-decomposed into interval colorable subgraphs in such a way that at each vertex the numbers of incident edges, in any pair of subgraphs, differ by at most one.*

Using different interval colorable graphs like stars and (union of) cycles, other bounds for specific cases were established.

THEOREM 33 ([7, Thm. 3.7 & Prop. 3.8]).

- *If G is an Eulerian bipartite graph, then $\theta_{\text{int}}(G) \leq \lceil \Delta(G)/4 \rceil$.*
- *If G is a bipartite graph with parts X and Y, then $\theta_{\text{int}}(G) \leq \min\{\Delta(X), \Delta(Y)\}$ where $\Delta(X)$ is the largest degree for a vertex in X.*

Note that the latter bound can be helpful if for one of the parts the largest degree is much smaller than the maximum degree.

For the biregular case, using matchings and the first result in Theorem 20, it was also shown that:

Theorem 34 ([7, Prop. 3.12]).

- *If G is $(3, 3r)$-biregular ($r \geq 2$), then $\theta_{\text{int}}(G) \leq 2$.*
- *If G is (k, kr)-biregular ($k \geq 4, r \geq 2$), then $\theta_{\text{int}}(G) \leq k - 2$.*

It was also noted that by Corollary 2 the *arboricity* of a graph G is an upper bound on $\theta_{\text{int}}(G)$ where the arboricity is defined as the smallest $k \in \mathbb{N}_0$ for which G can be edge-decomposed into forests. This is particularly good if the arboricity of the graph class is known to be small:

Theorem 35 ([7, Prop. 3.15]).

- *If G is a graph with q edges, then $\theta_{\text{int}}(G) \leq \sqrt{q/2}$.*
- *If G is planar, then $\theta_{\text{int}}(G) \leq 3$. If, additionally, G is triangle-free or outerplanar, then $\theta_{\text{int}}(G) \leq 2$.*

For some particular dense graphs, complete bipartite graphs were also considered for the decomposition to get some better bounds.

Theorem 36 ([7, Prop. 3.17, Prop. 3.18, Prop. 3.20 & Prop. 3.21]).

- *If G is a complete 4-partite graph, then $\theta_{\text{int}}(G) \leq 2$.*
- *If G is a complete r-partite graph, then $\theta_{\text{int}}(G) \in \mathcal{O}(\log r)$.*
- *For any $n, r \in \mathbb{N}$ ($r \geq 2$),*

$$\theta_{\text{int}}(K_{n*r}) = \begin{cases} 1, & \text{if } nr \text{ is even,} \\ 2, & \text{if } nr \text{ is odd.} \end{cases}$$

- *For any $n, r \in \mathbb{N}$ ($r \geq 2$), $\theta_{\text{int}}(K_{n*r,nr}) \leq 3$. Moreover, if nr is even, then $\theta_{\text{int}}(K_{n*r,nr}) = 1$.*

5. Variants of interval colorability

Apart from the relaxation using the fairly new notion of interval thickness, the concept of interval colorability has been generalized in various other ways. We will discuss some of those here that come from relaxing the conditions on the coloring.

5.1. One-sided interval colorings. Very early on, Asratian and Kamalian already considered in [9] and [10] the following relaxation of the problem, where in the bipartite setting you only want "no gaps" for one of the parts:

Definition 14. *Let G be a bipartite graph with parts A and B. We call an edge coloring $c\colon E(G) \to [t]$ an A-sided t-interval coloring if $P_c(A)$ is an interval for all $a \in A$ and $c(E(G)) = [t]$. Analogous to $w(G)$, we define $w_A(G)$ to be the smallest t for which an A-sided t-interval coloring exists.*

With this relaxation, it is actually always possible to find such a coloring. More specifically, one can find a coloring using $|E(G)|$ colors by simply coloring the edges of $a_1 \in A$ $1, \ldots, \deg(a_1)$, then coloring the edges of $a_2 \in A \setminus \{a_1\}$ $\deg(a_1) + 1, \ldots, \deg(a_1) + \deg(a_2)$, etc.

Hence, if we were to define $W_A(G)$ analogously to $W(G)$, it is clear that $W_A(G) = |E|$ for every bipartite graph $G = (A \cup B, E)$. Furthermore, the analogous object to a spectrum for A-sided intervals turns out to always be interval:

Theorem 37 ([10, Thm. 3]). *Let $G = (A \cup B, E)$ be a bipartite graph. Then for every $t \in [w_A(G), |E|]$ there exists an A-sided interval t-coloring of G.*

As we still require an A-sided interval coloring to be proper, it is clear that $w_A(G) \geq \Delta(G)$. However, determining $w_A(G)$ is still $\mathcal{NP}$-complete, which is evident given that deciding whether a $(3, 6)$-biregular graph is 6-interval colorable is $\mathcal{NP}$-complete (see Theorem 29).

Nevertheless, upper bounds on $w_A(G)$ have been obtained and sufficient conditions for $w_A(G) = \Delta(G)$ were found.

Theorem 38 ([17, Thm. 2.4]). *If $G = (A \cup B, E)$ is a bipartite graph, then*

$$w_A(G) \leq \frac{\Delta(G)^2(\Delta(G) + 1)}{2}.$$

Theorem 39 ([10, Col. 6]). *Let $G = (A \cup B, E)$ is a bipartite graph. If $\deg(a) \geq \deg(b)$ for every edge $ab \in E$ where $a \in A, b \in B$, then $w_A(G) = \Delta(G)$.*

Further results such as other upper and lower bounds on $w_A(G)$ can be found in [10], [39], [21] and [17].

5.2. Near-interval colorings. One possible generalization is to allow for each interval at most one gap. This leads to the concept of near-interval colorings, which were formally introduced in [55] as $(t, 1)$-*interval colorings.*

Definition 15 (Near-interval coloring / $(t, 1)$-interval coloring). *A proper coloring $c\colon E(G) \to [t], c(E(G)) = [t]$, of G is called an t-near interval coloring if for each $v \in V(G)$ either $P_c(v)$ is an interval or there exists $n(v) \in \mathbb{N}$ such that $P_c(v) \cup \{n(v)\}$ is an interval. Graphs for which there exists a t-near-interval coloring for some t are called near-interval colorable.*

We also note that both of these colorings can be obtained in polynomial time.

Unlike with interval colorability, not every near-interval colorable graph is Class 1. Indeed, the $(\Delta + 1)$-edge coloring of a Δ-regular graph is always also a $(\Delta + 1, 1)$-interval coloring which also includes all cycles and complete graphs.

However, it is also the case that not every bipartite graph, in particular not every Class 1 graph is near-interval colorable. More specifically, based on the non-interval colorable, bipartite graph defined by Malafiejski, it was shown that the generalized *Malafiejski rosette* given in Figure 7 is not near-interval colorable in [55].

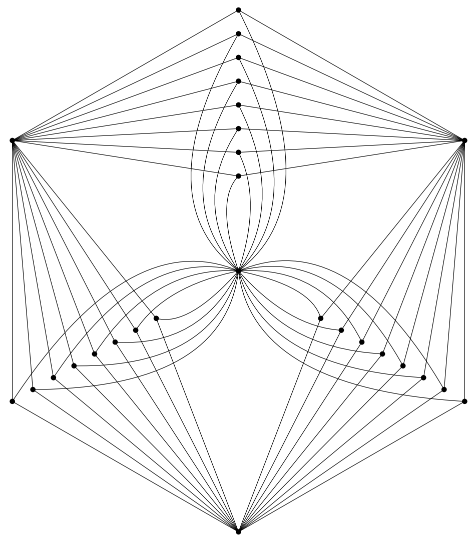

FIGURE 7. Malafiejski's rosette that is not near-interval colorable

More properties can be found in [55].

As the authors in [20] note, this relaxation is useful in the light of Conjecture 1: While for $n \geq 4$, the question whether $(n-1, n)$-biregular graphs are interval colorable is still open, it follows directly from Kőnig's Edge Coloring Theorem / Theorem 2 that $(n-1, n)$-biregular graphs are always near-interval colorable. Generally, it is clear that if a bipartite graph (or any Class 1 graph) G has minimum degree $n-1$ and maximum degree n, then G is near-interval colorable.

Focusing on the biregular case, it was also proven that:

THEOREM 40 ([20, Col. 3.2 & Thm. 3.3]).

- *Every* $(3,5)$*-biregular graph has a* 6*-near-interval coloring.*
- *Every* $(4,6)$*-biregular graph has a* 7*-near-interval coloring.*

The latter of the results is actually best possible: A 6-edge coloring c of a $(4,6)$-biregular graph $G = (A \cup B, E)$ is a 6-near-interval coloring if and only if $\{1,6\} \not\subseteq P_c(a)$ for all $a \in A$. As G is $(4,6)$-biregular, we have that $4|A| = |E| = 6|B|$, so $|A| = 3k$ and $|B| = 2k$ for some $k \in \mathbb{N}$. As the number of edges colored $i \in \{1,6\}$ is $|B| = 2k$, the pigeonhole principle implies that there is $a \in A$ with $\{1,6\} \subseteq P_c(a)$. So, no 6-edge coloring can be a near-interval coloring which also shows $w(G) > 6$ for every $(4,6)$-biregular graph G.

5.3. Cyclic interval colorings. Another variant that was introduced in [63] is that of *cyclic interval colorings*:

Definition 16 (Cyclic interval coloring). *A proper coloring* $c\colon E(G) \to [t], c(E(G)) = [t]$, *of* G *is called a cyclic* t*-interval coloring if for each* $v \in V(G)$ $P_c(v)$ *is an interval or* $[t] \setminus P_c(v)$ *is an interval.*

This variant is motivated by considering schedules with only 0-1-operations where the schedule needs to be repeated multiple times. More specifically, rephrasing it in the context of open shops with only 0-1-operations (see [47, p. 226]), it means that given an open shop instance $\mathbb{P}$, we say that the schedule S is a *cyclic compact open shop schedule* if the following two conditions hold:

- For each job there is a single interval between 0 and $C_{\max}(S)$ in which the job is processed, or a single interval between 0 and $C_{\max}(S)$ where the job is *not* processed.
- For each machine there is a single interval between 0 and $C_{\max}(S)$ in which the machine is occupied, or a single interval between 0 and $C_{\max}(S)$ where the machine is idle.

Clearly, if S is a cyclic compact open shop schedule, then the number of gaps during the repeated execution of the schedule is fairly small and in some cases as good as if S is a compact open shop schedule. The authors in [46] concretely state that these kinds of schedules appear in some industrial applications such as metallurgy or chemistry.

Note that every $(t, 1)$-interval coloring is a cyclic t-interval coloring.

With a similar approach as in the proof of Theorem 3, it is clear that every bipartite, interval colorable graph G has a cyclic $\Delta(G)$-interval coloring. However, it was shown in [51] that the converse does not hold. More specifically, *Malafiejski's rosettes*[4] were given as a counterexample to the other direction.

One open conjecture for this variant is the following:

Conjecture 2 ([20, Conj. 4.5]). *Every* (a, b)*-biregular graph has a cyclic* $\max\{a, b\}$*-interval coloring.*

A positive answer of this was given for $(a, b) = (4, 8)$.

Theorem 41 ([20, Thm. 4.6]). *Every* $(4, 8)$*-biregular graph has a cyclic* 8*-interval coloring.*

As the other results in [20], the proof of this can be turned into a polynomial algorithm to obtain the coloring.

Recall that this is not the case for interval colorability: By Theorem 29, determining whether a $(4, 8)$-biregular graph has an 8-interval coloring is $\mathcal{NP}$-complete and by Lemma 8 there is no such interval coloring if the graph doesn't contain a 4-regular subgraph that covers all vertices of degree 8.

A positive answer for $(a, b) = (3, 5)$ is also given in [19]:

[4] A generalization of Malafiejski's counterexample for a bipartite, non-interval colorable graph.

Theorem 42 ([19, Thm. 2.1]). *Every* $(3,5)$*-biregular graph has a cyclic 6-interval coloring.*

Characterizations and sufficient conditions were also given in [19] for a $(3,5)$-biregular graph to have a cyclic 5-interval coloring. Similar to the authors of [11] conjecturing that every $(3,4)$ satisfies the condition given in Theorem 17, the authors of [19] also believe that the characterization given in the paper is actually satisfied by all $(3,5)$-biregular graphs.

It is also interesting to note that later a more general pattern could be established:

Theorem 43 ([6, Thm. 2.10]). *If every* (a,b)*-biregular* $(a < b)$ *graph has a cyclic* b*-interval coloring and* $\gcd(a, b-1) = 1$*, then every* $(a, b-1)$*-biregular graph has a cyclic* b*-interval coloring.*

So, the fact that all $(3,6)$-biregular graphs have 6-cyclic interval colorings, which follows from them being interval colorable (see Theorem 20) and having maximum degree 6, implies Theorem 42. Similarly, we can conclude that every $(4,7)$-biregular graph has an 8-cyclic interval coloring by Theorem 41.

Concerning computational complexity, it is shown in [46] that determining whether a bipartite graph has a cyclic interval coloring is $\mathcal{NP}$-complete:

Theorem 44 ([46, Col. 2]). *Deciding whether a bipartite graph has a cyclic interval coloring is* $\mathcal{NP}$*-complete.*

The broad idea is to reduce the problem of determining whether a bipartite graph is interval colorable to the problem above. Since the former problem is $\mathcal{NP}$-complete (see [62]), the latter problem is also $\mathcal{NP}$-complete[5].

Examples for bipartite graphs without cyclic interval colorings are given in [6, 51, 38]. General methods to construct graphs with no cyclic interval colorings are studied in [58]. Apart from that, in [58] Petrosyan and Mkhitaryan also looked at general properties of such colorings and also for which t complete graphs, complete bipartite graphs, tripartite graphs and hypercubes have cyclic t-interval colorings.

Generally, it was shown that:

Theorem 45 ([6, Thm. 4.1]). *All complete multipartite graphs are cyclic interval colorable.*

This was first conjectured in [58].

For sufficient conditions when a bipartite graph or outerplanar graph has a cyclic interval coloring, see [6].

Bounds on the minimum number and maximum number, for which a graph is cyclic interval colorable, can be found in [58, 18].

[5] Clearly, the problem is in $\mathcal{NP}$.

5.4. Interval colorings on hypergraphs. It is interesting to note that in some ways the concept of interval colorability was even prior to its now coined named studied for hypergraphs. To be more specific, the decision problem 2-Dimensional Consecutive Sets (see [24, p. 230]) was studied:

Given a family $\mathcal{M} \subseteq 2^X$ of subsets of a finite ground set X and $k \in \mathbb{N}$, does there exist a partition $X_1 \uplus X_2 \uplus \ldots \uplus X_k = X$ such that

(i) $|M \cap X_i| \leq 1$ for all $M \in \mathcal{M}, i \in [k]$,

(ii) every $M \in \mathcal{M}$ is contained in th union $X_i \cup X_{i+1} \cup \cdots \cup X_{i+|M|-1}$ for a suitable $i \leq k$?

This problem was motivated by finding efficient file organizations and shown in [49] to be $\mathcal{NP}$-complete by reducing Graph 3-Colorability ([24, p. 191]) to it. This holds even under the restriction $|M| \leq 5$ for all $M \in \mathcal{M}$.

In graph theoretic terms, one may think of X as the vertex set and $\mathcal{M}$ as the set of hyperedges. In other words, the decision problem is equivalent to finding a proper[6] k-vertex coloring such that the colors of vertices incident to M form an interval for all hyperedges $M \in \mathcal{M}$.

To get from "interval vertex colorings" to interval colorings of hypergraphs (where we naturally extend the definition of interval colorability), we can define for a graph $\mathcal{G} = (X, \mathcal{M})$ the *dual* $\mathcal{G}^* = (X^*, \mathcal{M}^*)$ as follows:

$$X^* \coloneqq \mathcal{M}, \mathcal{M}^* \coloneqq \{M_x \colon x \in X\} \text{ where } M_x \coloneqq \{M \in \mathcal{M} \colon x \in M\}.$$

It is not hard to see that $(\mathcal{G}^*)^* \simeq \mathcal{G}$. Furthermore, a hypergraph $\mathcal{G}$ is k-interval colorable if and only if its dual $\mathcal{G}^*$ possesses a k-partition as desired in 2-Dimensional Consecutive Sets.

While hypergraphs are otherwise rarely considered in the study of interval colorings, doing so would also in some ways be of practical nature. In particular, one may consider Example 1 but instead of talks between only two people, we generalize the talks to conferences where an arbitrary size of people want to attend. This can then naturally be modeled by a hypergraph where the vertices are still the set of people but the hyperedges, which correspond to the conferences, are now the set of people attending a particular conference. Finding an interval coloring for this hypergraph would then again give us a compact schedule for the conferences.

6. Deficiency results

As we have seen, there are various ways to relax the conditions of an interval coloring. However, we have seen that some of them are still somewhat restrictive. For example, we know that not all bipartite graphs are cyclic interval colorable and hence also not near-interval colorable (see [6, 51, 38]). Still, on the application side, simply ignoring those instances isn't a very practical option. So, preceding the notion of interval thickness, the *deficiency* of a graph was studied: In the context of open shops for example, instead of just saying that no ideal schedule exists, it is more helpful to find for a given open shop instance (or in general any instance we have considered to schedule) a schedule that *minimizes* the waiting time between operations of a job and the idle time between operations on machines (or in general the number of "gaps").

[6]Here, we mean by proper that no two vertices sharing the same color are both contained in the same hyperedge.

From a theoretical standpoint, this is also interesting as we are essentially looking for a measure for "how close" a graph is to being interval colorable.

This leads to the following definition which was first introduced in [29, 30]:

Definition 17 (Deficiency). *For a given graph G, the deficiency of G is defined as the minimum number of pendant edges whose attachment to G makes it interval colorable. We denote it by* $\operatorname{def}(G)$.

Obviously, a graph is interval colorable if and only if its deficiency is zero. It is also clear that, in the open shop with only 0-1-operations setting, the deficiency is equal to the sum of the waiting times between operations of a job and the idle times between operations on machines.

However, unlike with the interval thickness, lower bounds on the deficiency are established, even for bipartite graphs (see [30, 29]). In particular, building on the counterexample by Hertz (see [56, 30]), it was shown in [30], that there exists a sequence of planar bipartite, connected graphs G_k for which $\lim_{k\to\infty}(\operatorname{def}(G_k)/|G_k|) = 1$, i.e. where the deficiency is asymptotically equal to the number of vertices.

It was also shown that for an r-regular graph G with an odd number of vertices, $\operatorname{def}(G) \geq r/2$ (see [29]).

This led the author to the following question:

Open Problem ([30]). *If we define*

$$\operatorname{def}(n) := \max\{\operatorname{def}(G)\colon G \textit{ is a graph}, |V(G)| \leq n.\},$$

it follows that $\operatorname{def}(n) \in \Omega(n)$. *On the other hand, we can show that* $\operatorname{def}(G) < n^2$ *for any graph G: By Vizing's Theorem / Theorem 1, every graph has a $(\Delta(G)+1)$-edge coloring c. We can extend c to a $(\Delta(G)+1)$-interval coloring of a supergraph of G, if we "fill the gaps" of $P_c(v)$ for every $v \in V(G)$ by adding suitably many pendant edges. Clearly, at most $(\Delta(G)-1)$ edges are needed per vertex, so* $\operatorname{def}(G) \leq (\Delta(G)-1)n < n^2$. *So,* $\operatorname{def}(G) \in \mathcal{O}(n^2)$. *However, this bound seems far from tight, so the question is:*

What is the order of growth of $\operatorname{def}(n)$*?*

The question was somewhat investigated for regular graphs:

Theorem 46 ([61]). *Let R_k be the set of k-regular graphs, $k \in \mathbb{N}$, and define*

$$s_k := \sup_{G\in R_k} \frac{\operatorname{def}(G)}{|V(G)|}.$$

- *We have for $k \in \{1,2,3,4\}$*

$$s_k = \begin{cases} 0, & k=1 \\ \frac{1}{3}, & k=2 \\ \frac{1}{5}, & k=3 \\ \frac{2}{5}, & k=4. \end{cases}$$

- *We have for* $k \equiv 1 \pmod 2$
$$\frac{k-1}{k^2+1} \le s_k \le \frac{k-1}{k+1}.$$

- *We have for* $k \equiv 0 \pmod 2$
$$\frac{k}{2k+2} \le s_k \le \frac{k-1}{k+1}.$$

This shows that $\max\{\mathrm{def}(G)\colon G \text{ is a regular graph}, |V(G)| \le n.\} \in \Theta(n)$. Looking at the results for $k \in \{2,3,4\}$, however, the following problem was posed:

Open Problem. *Let n be the fewest number of vertices on which a k-regular graph with nonzero deficiency exists. Is it always true that there exists a k-regular graph on n vertices with deficiency $s_k n$?*

Note that the answer is positive for $k = 2$ by considering a triangle and for $k \in \{3,4\}$ the answer also turns out to be positive (see [61]).

The deficiency of certain graph classes was also determined. In particular the deficiency of cycles, complete graphs, wheels and the so-called *broken wheels* are given in [29] and the deficiency of *general Sevastianov rosettes*[7] and *generalized Hertz graphs*[8] in [15] and [14] respectively.

Lastly, we note that for practical purposes Bodur and Luedtke propose several integer linear programming formulations to the problem of finding $\mathrm{def}(G)$ in [13]. Other such formulations can be found in [47, ch. 9.5.2].

Naturally, deficiency was generalized for cyclic interval colorings.

Definition 18 (Cyclic deficiency, [4]). *For a given graph G, the cyclic deficiency of G is defined as the minimum number of pendant edges whose attachment to G makes it cyclic interval colorable. We denote it by* $\mathrm{def}_c(G)$.

Clearly, as every interval coloring is also a cyclic interval coloring, $\mathrm{def}_c(G) \le \mathrm{def}(G)$. As with deficiency, $\mathrm{def}_c(G) = 0$ if and only if G is cyclic interval colorable.

However, there are many cases where the inequality is far from tight, i.e. $\mathrm{def}_c(G) \ll \mathrm{def}(G)$. For example, for the general Sevastianov rosettes, the deficiency grows linearly in the number of vertices (see [30, 15]) while its cyclic deficiency was shown to be zero [4].

More generally, the following was shown:

Theorem 47 ([4, Thm. 3.2]). *For any $m, n \in \mathbb{N}, m \le n$, there exists a connected graph G with bounded maximum degree such that* $\mathrm{def}_c(G) = m$ *and* $\mathrm{def}(G) = n$.

So, not only can $\mathrm{def}(G) - \mathrm{def}_c(G)$ be arbitrarily large, $\mathrm{def}_c(G)$ can grow arbitrarily large even if the corresponding graphs have bounded maximum degree.

[7] A class bipartite graphs that generalizes Sevastianov's graph (see Definition 11).
[8] Those generalize the generalization given in [30] of Hertz's original graph.

In [4], they also gave several constructions for which the cyclic deficiency is large. For example, it was shown that $\operatorname{def}(G_k) = \operatorname{def}_c(G_k)$ for the Hertz graphs G_k as defined in [30], showing $\lim_{k\to\infty}(\operatorname{def}_c(G_k)/|V(G_k)|) = 1$. However, no good general upper bounds of $\operatorname{def}_c(G)$ are known, but they conjectured:

Conjecture 3 ([4, Conj. 5.4]). *For any graph G, $\operatorname{def}_c(G) \leq |V(G)|$.*

Formulated as in the open problem 6, the conjecture states $\operatorname{def}_c(n) \leq n$ for every $n \in \mathbb{N}$ where

$$\operatorname{def}_c(n) \coloneqq \max\left\{\operatorname{def}_c(G)\colon G \text{ is a graph}, |V(G)| \leq n.\right\}.$$

Sufficient conditions for when Conjecture 3 is true are also given in [4].

Chapter 5

Certain classes of biregular graphs and their interval thickness

In this chapter, we will look at classes of biregular graphs.

While the question whether every biregular graph is interval colorable is still unanswered with the smallest unknown case being $(3, 4)$-biregular graphs (see [20]), we will now focus on specific classes of biregular graphs.

1. Incidence graphs

Definition 19 (Incidence graph). *The incidence graph $G_{n,k}$ with $n \geq k > 0$ is a bipartite graph with parts $A = X$ with $|X| = n$ and $\mathcal{B} = \binom{X}{k}$ and edge set*

$$E(G_{n,k}) \coloneqq \{aB \colon a \in A, B \in \mathcal{B}, a \in B\}.$$

Remark 4. *Indeed, every $G_{n,k}$ is biregular. From the definition, it follows:*

- *For all $a \in A$ we have* $\deg(a) = \binom{n-1}{k-1}$.
- *For all $B \in \mathcal{B}$ we have* $\deg(B) = k$.

It is also clear that $\theta_{\mathrm{int}}(G_{n,1}) = 1, \theta_{\mathrm{int}}(G_{n,n}) = 1$. For $G_{n,n-1}$ we note that the graph is then $(n-1)$-regular, hence a regular Class 1 graph that by Theorem 2 is interval colorable.

Lemma 11. $\theta_{\mathrm{int}}(G_{n,2}) = 1$ *for $n \geq 2$.*

Proof. W.l.o.g. $X = [n]$. We will explicitly define a $(2n-1)$-interval coloring: Define $c \colon E(G_{n,2}) \to \{2, 3, \ldots, 2n-1\}$ by

$$c(a, \{a, a'\}) \coloneqq \begin{cases} a + a' & \text{if } a' < a, \\ a + a' - 1 & \text{if } a < a'. \end{cases}$$

Clearly, looking at the edges for fixed $B \in \mathcal{B}$, the incident edges of B have consecutive colors. Now, fix $a \in A$. The set of edges incident to a is

$$E_a = \{\{a, 1\}, \ldots, \{a, a-1\}, \{a, a+1\}, \ldots, \{a, n\}\}.$$

We have that $c(\{a, i\}) = a + i$ if $i < a$ and $c(\{a, i\}) = a + i - 1$ if $i > a$. Thus, the colors used on E_a are $a+1, a+2, \ldots, 2a-1, 2a, \ldots, a+n-1$, respectively. So, the colors form an interval. This shows the claim. □

Unfortunately, we weren't able to make progress in the case $k \geq 3$. Also, note that Theorem 20 implies Lemma 11, but we hope that the explicit coloring given may lead to progress for $k \geq 3$.

2. Layers of the Boolean lattice

Definition 20. *For $n \in \mathbb{N}$ and $0 \leq k < n$ we define $J_n(k, k+1)$ as the bipartite graph with parts $\mathcal{A} = \binom{[n]}{k}, \mathcal{B} = \binom{[n]}{k+1}$ and $E(J_n(k, k+1)) := \{AB : A \in \mathcal{A}, B \in \mathcal{B}, A \subseteq B\}$.*

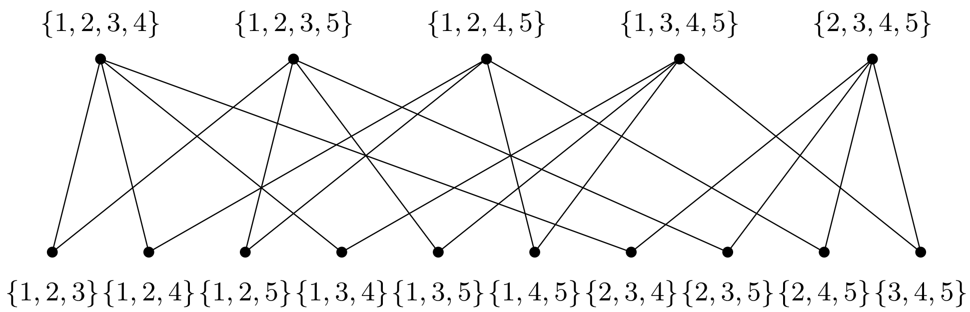

Figure 1. $J_5(3, 4)$

Note that by definition $\binom{X}{0} = \{\emptyset\}$ for arbitrary set X.

Remark 5. *Indeed, $J_n(k, k+1)$ are biregular graphs as*

- $\deg(A) = n - k$ *for* $A \in \binom{[n]}{k}$,
- $\deg(B) = k + 1$ *for* $B \in \binom{[n]}{k+1}$.

Clearly, $J_n(1, 2)$ is naturally isomorphic to $G_{n,2}$ by identifying singleton sets with their element. From Lemma 11 we therefore get that $\theta_{\text{int}}(J_n(1, 2)) = 1$ for $n \geq 2$. However, we can go even further.

Definition 21. *For a finite set of natural numbers $X \subset \mathbb{N}$, let $s(X) := \sum_{x \in X} x$ be the sum of X and define the order $\text{ord}_X(x)$ for $x \in X$ to be*

$$\text{ord}_X(x) := |\{y \in X : y \leq x\}|.$$

For $n > k \geq 0$, we then define the coloring c_n of $J_n(k, k+1)$ as follows:

$$c_n(A(A \cup \{b\})) := s(A) + \text{ord}_{[n] \setminus A}(b) \text{ for } A \in \binom{[n]}{k}, b \in [n] \setminus A.$$

Theorem 48. *c_n (as defined in Definition 21) is an interval coloring of $J_n(k, k+1)$ for $n > k \geq 0$. More specifically, we have for $A \in \binom{[n]}{k}$ and $B \in \binom{[n]}{k+1}$*

$$P_{c_n}(A) = [s(A) + 1, s(A) + n - k] \text{ and } P_{c_n}(B) = [s(B) - k, s(B)].$$

In particular, for every $k \in \mathbb{N}_0$ and $n > k$ we have $\theta_{\text{int}}(J_n(k, k+1)) = 1$.

Proof. Let $n > k \geq 0$. Note that if $A \in \mathcal{A}$ and $B \in \mathcal{B}$ are adjacent, then $B \setminus A$ contains a single element, so c_n is a well-defined. Also, note that if the palettes of the vertices are as claimed, then c_n is automatically a proper edge coloring as $|P_{c_n}(v)| = \deg(v)$ for all $v \in J_n(k, k+1)$.

1. Let $A \in \mathcal{A}$. For each $i = 1, \ldots, n-k$, there is exactly one element $b \in [n] \setminus A$ with $\text{ord}_{[n] \setminus A}(b) = i$. Therefore,

$$P_{c_n}(A) = [s(A) + 1, s(A) + n - k].$$

2. Let $B \in \mathcal{B}$. Order the elements $b_1, \ldots, b_k, b_{k+1}$ in B such that

$$b_1 < b_2 < \cdots < b_k < b_{k+1}.$$

Observe that $\mathrm{ord}_{[n]\setminus(B\setminus\{b_j\})}(b_j) = |[b_j] \setminus \{b_1, \ldots, b_{j-1}\}| = b_j - (j-1)$. Therefore, it follows $P_{c_n}(B) = [s(B) - k, s(B)]$ since for $j = 1, \ldots, k$

$$c_n((B \setminus \{b_j\})B) = s(B \setminus \{b_j\}) + b_j - (j-1) = s(B) - (j-1).$$

So, c_n is an interval coloring of $J_n(k, k+1)$. This concludes the proof. $\square$

Note that the coloring defined in Definition 21 is a direct generalization of the coloring in Lemma 11. An immediate result of this is that the Hasse diagram of a union of "consecutive" layers of the Boolean lattice is interval colorable.

Corollary 7. *For integers $0 \leq d \leq u < n$, we have*

$$\theta_{\mathrm{int}}(J_n(d, d+1) \cup J_n(d+1, d+2) \cup \cdots \cup J_n(u, u+1)) = 1.$$

Proof. Let $G \coloneqq J_n(d, d+1) \cup J_n(d+1, d+2) \cup \cdots \cup J_n(u, u+1)$. Clearly, every edge of G is contained in exactly one of the $J_n(l, l+1), d \leq l \leq u$. So, we may color edges in $J_n(l, l+1)$ as we would according to Definition 21 for all $d \leq l \leq u$. As $B \in V(G)$ with $|B| = d$ have all their incident edges in $J_n(d, d+1)$, the colors of B's incident edges form an interval. The same holds for $B \in V(G)$ with $|B| = u + 1$.

So, consider $B \in V(G)$ with $d + 1 \leq |B| \leq u$. Every edge incident to B is either in $J_n(|B| - 1, |B|)$ or $J_n(|B|, |B| + 1)$. By Theorem 48, the colors of those in $J_n(|B| - 1, |B|)$ form the interval $[s(B) - (|B| - 1), s(B)]$ while the colors of those in $J_n(|B|, |B| + 1)$ form the interval $[s(B) + 1, s(B) + n - |B|]$. On the whole, the colors of the edges incident to B therefore form the interval $[s(B) - (|B| - 1), s(B) + n - |B|]$. $\square$

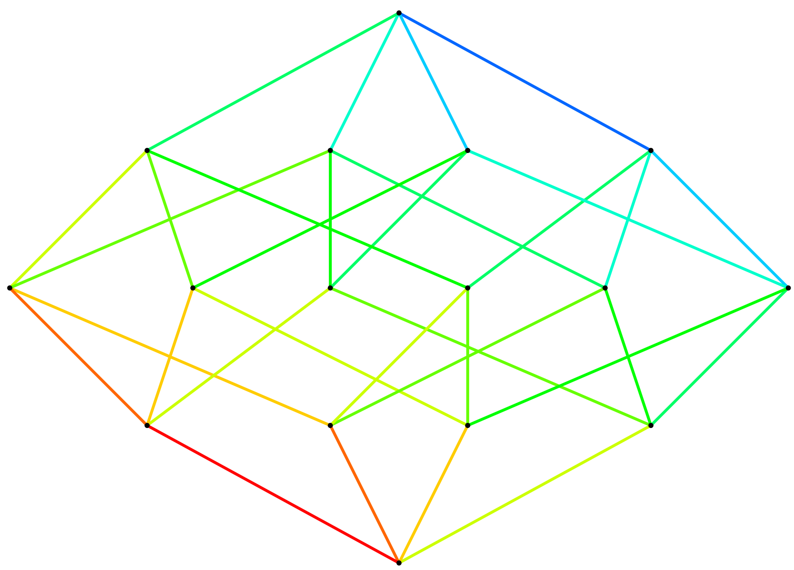

Figure 2. Coloring of Q_4 from Cor. 7 with $\emptyset$ at the bottom and $[4]$ at the top. The colors go from red to blue. The subsets in each layer are ordered lexicographically.

Remark 6. *Corollary 7 yields for $d = 0, u = n-1$ a peculiar interval coloring c of the Hasse diagram of the whole Boolean lattice, which is isomorphic to the n-dimensional hypercube Q_n: We observe that $P_c(\emptyset) = [1, n], P_c([n]) = [s([n]) - (n-1), s([n])]$ and*

$$\forall\, \emptyset \neq B \subset [n]\colon P_c(B) = [s(B) - (|B| - 1), s(B) + n - |B|].$$

Note that $s(B)$ is minimal if $B = [|B|]$ and maximal if $B = [n - |B| + 1, n]$. So,

$$1 \leq s([|B| - 1]) + 1 = s([|B|]) - |B| + 1 \leq s(B) - (|B| - 1),$$

giving us that 1 is overall the smallest "used" color by c. Similarly,

$$\begin{aligned} s(B) + n - |B| &\leq s([n - |B| + 1, n]) + n - |B| \\ &= s([n]) - s([n - |B|]) + (n - |B|) \\ &= s([n]) - s([n - |B| - 1]) \\ &\leq s([n]), \end{aligned}$$

giving us that $s([n]) = n(n+1)/2$ is overall the largest "used" color by c. As Q_n is connected, c must therefore be an $n(n+1)/2$-interval coloring of Q_n by Lemma 2, with $n(n+1)/2$ incidentally being $W(Q_n)$ (see Theorem 13).

This was actually first conjectured by Petrosyan in [52, Conj. 21] before it was finally proven by Petrosyan et al. in [57].

We conclude this chapter with some conjectures.

Conjecture 4. $\theta_{\text{int}}(J_n(i, j)) = 1$ *for every* $n \in \mathbb{N}$ *and* $0 \leq i < j \leq n$.

Conjecture 5. $\theta_{\text{int}}(G_{n,k}) = 1$ *for every* $0 < k \leq n$.

Note that the former conjecture implies the latter as $G_{n,k}$ is naturally isomorphic to $J_n(1, k)$ by identifying elements with the corresponding singleton set.

It would also be interesting to look at $w(G)$ and $W(G)$ for G as in Corollary 7 since the spectrum seems "weirdly nice" in the special case of Q_n (see Theorem 13).

Chapter 6

Graphs with arbitrarily many gaps in their spectrum

In this chapter, we will take a closer look at the spectrum. Recall that it is defined as the set of $t \in \mathbb{N}$ for which our given graph is t-interval colorable (see Definition 6).

Suggested by the results of the paper [36], it was an open question for a while whether for every bipartite graph the spectrum is an interval itself as in the case of complete bipartite graphs or trees.

The answer to that question turned out to be negative: In [62], Sevastianov gave an example for a bipartite graph with a *gap* in its spectrum, i.e. whose spectrum can't be represented by an interval.

We will extend this result now. For that, we will use a special case of the gadget graph introduced in [62] as a black box and start by reviewing its most important properties: Given an interval $I = [a, b] \subset \mathbb{N}, a \leq b$, and even $D \geq b$, Sevastianov defined the bipartite graph[1] $G(\Theta)$ where Θ is defined as the pair $\Theta = (I, D)$. It contains the edge[2] $e(\Theta) = e_{G(\Theta)} \in G(\Theta)$ and a bundle $\pi(\Theta) = \pi_{G(\Theta)}$ of edges. From Figure 1, we can see that $G(\Theta)$ is also planar.

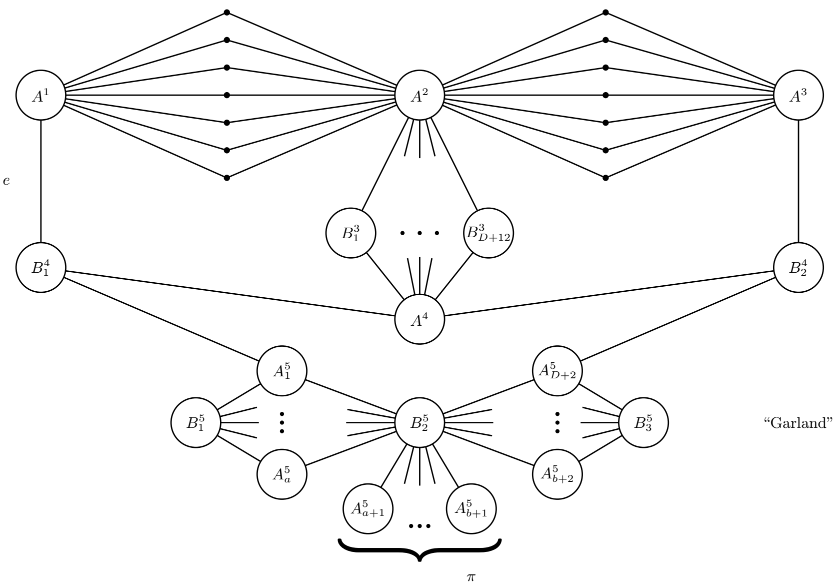

FIGURE 1. $G(\Theta)$ with some of the vertex labelings from the paper

From the lemma given in [62], the following holds:

[1]In the original paper, it would be denoted by $G_{1,\{I\}}$ with the parameter D implicitly set.
[2]Originally in the paper denoted by a_1.

LEMMA 12 ([62]). *$G(\Theta)$ is interval colorable. Moreover, for any T-interval coloring φ the following holds:*

a) the number of colors is $T = D + 26$;

b) $\varphi(e(\Theta)) \in \{c_0, T + 1 - c_0\}$ where $c_0 = 9$;

c) $\varphi(\pi(\Theta))$ is an interval; moreover, if $\varphi(e(\Theta)) = c_0$, then $\varphi(\pi(\Theta)) = c_1+[a, b]$, where $c_1 = 13$.

Generalizing Sevastianov's approach of taking many instances of the gadget graph and modifying them, we obtain the following result:

THEOREM 49. *For every $k, d \in \mathbb{N}$ there exists a planar bipartite graph G such that* $\operatorname{spec}(G)$ *can't be written as the union of k intervals and if written as the disjoint union of non-neighboring[3] intervals*

$$[a_1, b_1], \dots, [a_l, b_l] \subset \mathbb{N}, l > k,$$

with $a_1 \leq b_1 \leq \cdots \leq a_l \leq b_l$, then $a_{i+1} - b_i \geq d$ for $i \in [l - 1]$.

PROOF. W.l.o.g. is k even and d odd. Let $d' := d+k-1$. Set $\Theta := ([1, k], k(d'+1))$ and $\Theta_i := (\{1\}, id')$ for $i \in [k]$. Consider $G(\Theta), G(\Theta_1), \dots, G(\Theta_k)$. Let $uu_1, \dots, uu_k$ denote the vertices in $\pi(\Theta)$ and denote the single leaf and pendant edge in $\pi(\Theta_i)$ by w_i and v_iw_i respectively. We construct G as the union of $G(\Theta), G(\Theta_1), \dots, G(\Theta_k)$ where we identify the following vertices and otherwise consider them vertex-disjoint:

- $w_1 \simeq \cdots \simeq w_k \simeq u$,
- $u_i \simeq v_i$ for $i \in [k]$.

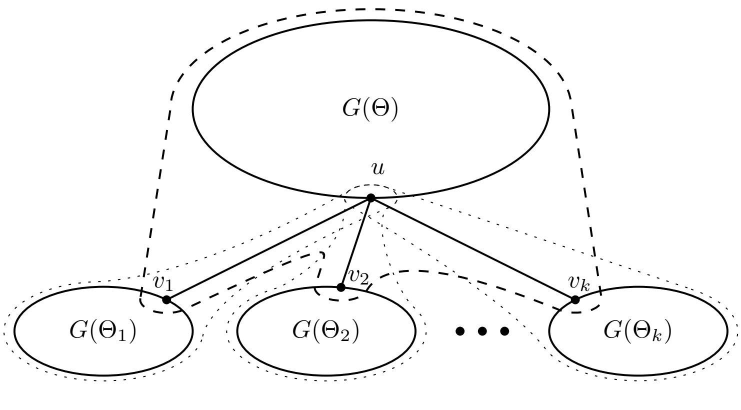

FIGURE 2. Sketch of G

Clearly, G is bipartite as every $G(\Theta), G(\Theta_1), \dots, G(\Theta_k)$ is bipartite. From Figure 2, we also see that G is planar. As G is connected, we know by Lemma 1 and 2 that G is t-interval colorable if and only if there exists an interval coloring of G using t colors. Let φ be a interval coloring of G using $t \in \mathbb{N}$ colors. Restricted onto $V(G(\Theta))$ or $V(G(\Theta_i))$, φ induces an interval coloring for $G(\Theta), G(\Theta_1), \dots, G(\Theta_k)$. On the other hand, we can construct an interval coloring of G by taking interval colorings of $G(\Theta), G(\Theta_1), \dots, G(\Theta_k)$ that match in the pendant edges. Note that by Lemma 12, each interval coloring of $G(\Theta_i)$ has two possibilites as to where the color of $e(\Theta)$

[3]We say that $[a, b]$ and $[c, d]$ with $a \leq b < c \leq d$ are non-neighboring if $b + 1 < c$.

is "located" in the interval of colors used for $G(\Theta_i)$, the same goes for $G(\Theta)$ and $e(\Theta)$. So, using Lemma 1 and 12, we may suitably shift φ such that $\varphi(E(G(\Theta))) = [1, k(d'+1)+26]$. Lemma 12 then implies that $\varphi(e(\Theta)) \in \{9, (k(d'+1)+26)+1-9\}$. W.l.o.g., let $\varphi(e(\Theta)) = 9$ be the case, otherwise start out with the interval coloring $-\varphi$ instead of φ. Therefore, Lemma 12 implies $\varphi(\pi(\Theta)) = [14, 13+k]$.

Fix $i \in [k]$ and let $\varphi(uu_i) = \varphi(v_i w_i) = c_i \in \varphi(\pi(\Theta))$. Again, using Lemma 12,

$$\varphi(G(\Theta_i)) = [1, id'+26] + (c_i - 13) \qquad = [c_i - 12, c_i + id' + 13], \text{ or}$$
$$\varphi(G(\Theta_i)) = [1, id'+26] + (c_i - (id'+26+1-13)) = [c_i - id' - 13, c_i + 12].$$

Note that the shifts are chosen such that c_i is either the 13-th color or the $(id' + 26 + 1 - 13)$-th color of $\varphi(G(\Theta_i))$ which are the only two possibilities for an interval coloring of $G(\Theta)$.

As $c_i - 12 \geq 1$ and $c_i + id' + 13 \leq k(d'+1)+26$, if the former case holds for all $i \in [k]$, we get that $\varphi(G) = [1, k(d'+1)+26]$. If the latter case holds for some $i \in [k]$, it implies that $[c_i - id' - 13, k(d'+1)+26] \subseteq \varphi(G)$. As

$$\begin{aligned} c_i - id' - 13 \geq 1 - id' \qquad & \geq (k - d') - id' \\ &= (13+k) - id' - d' - 13 \geq c_{i+1} - (i+1)d' - 13, \end{aligned}$$

$i \mapsto c_i - id' - 13$ is decreasing in i. So, $\varphi(G) = [c_{i_{\max}} - i_{\max}d' - 13, k(d'+1)+26]$ where $i_{\max}$ is the maximum $i \in [k]$ for which the latter case holds.

Therefore, the number of colors φ uses is

$$\begin{aligned} &(k(d'+1)+26) - (c_{i_{\max}} - i_{\max}d' - 13) + 1 \\ &= (i_{\max}+k)d' + k + 40 - c_{i_{\max}} \\ &\in [(i_{\max}+k)d' + k + 40 - (13+k), (i_{\max}+k)d' + k + 40 - 14] \\ &= [(i_{\max}+k)d' + 27, (i_{\max}+k)d' + 26 + k]. \end{aligned}$$

Clearly, for every $c \in [14, 13+k]$ and $i \in [k]$ there exists an interval coloring of G such that $i_{\max} = i$ and $c_{i_{\max}} = c$. Thus, we get

$$\operatorname{spec}(G) = \{\underbrace{kd' + (26+k)}_{=:a_1=:b_1}\} \cup \bigcup_{i=1}^{k} [\underbrace{(i+k)d' + 27}_{=:a_{i+1}}, \underbrace{(i+k)d' + (26+k)}_{=:b_{i+1}}].$$

We see that

$$a_{i+1} - b_i = (i+1+k)d' + 27 - ((i+k)d' + (26+k)) = d' - (k-1) = d$$

for $i \in [k]$. In particular, $\operatorname{spec}(G)$ can't be written as the union of k intervals. $\square$

Remark 7. *This result implies that for schedulings satisfying the "interval property", the makespan can be "rigid". More specifically, it might be the case that if we have a compact schedule S with makespan $C_{\max}(S)$ and consider the compact schedule S' with the next highest makespan $C_{\max}(S')$, then $C_{\max}(S) \ll C_{\max}(S')$ might be the case. This could be of practical importance in the following way: Let us revisit the Example 1 with the additional constraint that each talk needs to be held in a room with no two talks happening in the same room and the number of rooms being a constant $K \in \mathbb{N}$. This is equivalent to each color class containing at most K edges. Now, if every t-interval coloring φ of the meeting graph has a color class of size bigger than K, then we would have to resort to consider interval colorings using more than t colors where the next possible "candidate" for the number of colors might be much greater than t.*

Chapter 7

Improved upper bounds on the interval thickness of dense graphs

While the new upper general bounds given in [7] are better than naively taking the induced subgraphs obtained from color classes of an edge coloring of the graph G, it still gives us bounds in $\Theta(\Delta(G))$. Especially for dense graphs, there seems to be a lot of room of improvement: In this case, the maximum degree is roughly equal to the number of vertices in the graph, so those new bounds grow roughly linearly in the number of vertices for a dense graph.

In this section, we present better asymptotic results by approaching the problem using extremal methods. For that, we will first revise some definitions and important theorems.

Definition 22 ([22, 1]). *Let $\emptyset \neq X, Y \subseteq V(G)$ be disjoint vertex sets and $\varepsilon > 0$.*

- *We define $\|X, Y\|$ to be the number of edges between X and Y and the density $d(X, Y)$ of (X, Y) to be*
$$d(X,Y) \coloneqq \frac{\|X,Y\|}{|X|\,|Y|}.$$
- *The pair (X, Y) is an ε-regular pair or more precisely a (d, ε)-pair if we have $|d - d(A, B)| \leq \varepsilon$ for all $A \subseteq X, B \subseteq Y$ with $|A| \geq \varepsilon |X|, |B| \geq \varepsilon |Y|$ and $d = d(X, Y)$.*
- *For an ε-regular pair (X, Y), we define*
$$G[X,Y] \coloneqq G[X \cup Y] - E(G[X]) - E(G[Y])$$
to be the corresponding ε-regular bipartite graph.
- *For (X, Y), we let $\delta(X, Y) \coloneqq \delta(G[X, Y])$ and $\Delta(X, Y) \coloneqq \Delta(G[X, Y])$.*
- *We call an ε-regular pair (X, Y) super ε-regular if*
$$(d(X,Y) - \varepsilon)n \leq \delta(X,Y) \leq \Delta(X,Y) \leq (d(X,Y) + \varepsilon)n.$$
- *An ε-regular partition of the graph $G = (V, E)$ is a partition of the vertex set $V = V_0 \mathbin{\dot\cup} V_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} V_k$ with the following properties:*

  1. *$|V_0| \leq \varepsilon |V|$*
  2. *$|V_1| = |V_2| = \cdots = |V_k|$*
  3. *All but at most εk^2 of the pairs (V_i, V_j) for $1 \leq i < j \leq k$ are ε-regular.*

THEOREM 50 (Szemerédi's Regularity Lemma [22]). *For every $\varepsilon > 0$ and every integer $m \in \mathbb{N}$ there is an $M \in \mathbb{N}$ such that every graph of order at least m has an ε-regular partition $V_0 \cup \ldots \cup V_k$ with $m \leq k \leq M$.*

In [1], Alon et al. proved the following lemma:

LEMMA 13. *Let G be a super (d, ε)-regular graph with parts A and B, $|A| = |B| = n$, $d > 2\varepsilon$. Then G contains spanning k-factor, where $k = \lceil (d - 2\varepsilon)n \rceil$.*

However, as stated in the remark of their paper, the super (d, ε)-regularity can be relaxed by omitting the assumption on the maximum degree. Furthermore, note that the spanning k-factor in Lemma 13 is interval colorable by Theorem 2 and 8.
Thus we get:

COROLLARY 8. *Let G be a (d, ε)-regular graph with parts A and B, $|A| = |B| = n$, $d > 2\varepsilon$ and $(d(X, Y) - \varepsilon)n \leq \delta(G)$. Then G contains an interval colorable subgraph H of size at least $(d - 2\varepsilon)n^2$.*

The following lemma gives us a result for a general ε-regular pair.

LEMMA 14. *Let (X, Y) be a (d, ε)-regular pair with $d > 4\varepsilon$ and $|X| = |Y| = n$. Then, there are $X' \subseteq X, Y' \subseteq Y$ such that $|X'| = |Y'| > (1-\varepsilon)n$ and (X', Y') is a 3ε-regular pair with density d' where $d + \varepsilon \geq d' \geq d - \varepsilon$ such that $\delta(X', Y') \geq (d' - 3\varepsilon)|X'|$. Moreover, if $d - \varepsilon > 6\varepsilon$, then $G[X, Y]$ contains an interval colorable subgraph H of size at least $(d - 7\varepsilon)(n(1 - \varepsilon))^2$.*

PROOF. Let $\tilde{X} \coloneqq \{x \in X \colon\ |N(x) \cap Y| \geq (d - \varepsilon)|Y|\}$ and

$$\tilde{Y} \coloneqq \{y \in Y \colon\ |N(y) \cap X| \geq (d - \varepsilon)|X|\}.$$

First, note that $|\tilde{X}| > (1 - \varepsilon)|X|$: Assume otherwise. By ε-regularity, we have that $|d - d(X \setminus \tilde{X}, Y)| \leq \varepsilon$, which implies $d(X \setminus \tilde{X}, Y) \geq d - \varepsilon$. On the other hand,

$$d(X \setminus \tilde{X}, Y) < \frac{|X \setminus \tilde{X}|(d - \varepsilon)|Y|}{|X \setminus \tilde{X}||Y|} = d - \varepsilon. \lightning$$

Similarly, we have $|\tilde{Y}| > (1 - \varepsilon)|Y|$. Now, let $X' \subseteq \tilde{X}, Y' \subseteq \tilde{Y}$ such that

$$|X'| = |Y'| = \min\{|\tilde{X}|, |\tilde{Y}|\} > (1 - \varepsilon)|X| > \varepsilon n.$$

Note that the last inequality follows from $1 \geq d > 4\varepsilon$, which implies $1/4 > \varepsilon$.

Let $d' \coloneqq d(X', Y')$. By ε-regularity of (X, Y), we have

$$d + \varepsilon \geq d' \geq d - \varepsilon.$$

By definition of $\tilde{X}$ and $\tilde{Y}$, we have

$$\delta(X', Y') \geq (d - \varepsilon)n - \varepsilon n = (d - 2\varepsilon)n \geq (d' - 3\varepsilon)|X'|.$$

Finally, we show that (X', Y') is indeed a 3ε-regular pair: Consider $A \subseteq X', B \subseteq Y', |A| \geq 3\varepsilon|X'|$ and $|B| \geq 3\varepsilon|Y'|$. Observe that

$$3|X'| > 3(1 - \varepsilon)n > \frac{9}{4}n > n,$$

so $|A| > \varepsilon n$. Similarly, one may argue that $|B| > \varepsilon n$. Then, by ε-regularity of (X, Y), we obtain using the triangle inequality that

$$|d(X', Y') - d(A, B)| \leq |d(X', Y') - d(X, Y)| + |d(A, B) - d(X, Y)| \leq 2\varepsilon \leq 3\varepsilon.$$

Thus, (X', Y') is a 3ε-regular pair with density $d - \varepsilon \leq d' \leq d + \varepsilon$ and minimum degree $\delta(X', Y') \geq (d' - 3\varepsilon) |X'|$. Applying Corollary 8, we get the last result. $\square$

We now use this lemma and Szemerédi's Regularity Lemma / Theorem 50 to get a better bound for the interval thickness of dense graphs: By applying the Regularity Lemma, we can roughly partition the vertices of a sufficiently large graph into a bounded number of parts of the same size where the edges between any two parts behave "homogeneously" in the sense of ε-regularity. For those pairs of high enough density, we can then apply the previous Lemma to obtain those factors, which are interval colorable. The remaining number of edges will be very small.

THEOREM 51. *For every $\gamma > 0$, there exists $M \in \mathbb{N}$ such that every graph G contains a subgraph G' with $\theta_{\text{int}}(G') \leq M$ and $\|G\| - \|G'\| \leq \gamma |G|^2$. More specifically, for $\varepsilon > 0$ and $m \in \mathbb{N}$ with*

$$\frac{1}{2m} + \frac{21 + \varepsilon}{2} \cdot \varepsilon \leq \gamma,$$

we can choose M as in Szemerédi's Regularity Lemma / Theorem 50.

PROOF. W.l.o.g. we may assume $\gamma \leq 1/2$. Choose $\varepsilon > 0$ sufficiently small and $m \in \mathbb{N}$ sufficiently large such that

$$\frac{1}{2m} + \frac{21 + \varepsilon}{2} \cdot \varepsilon \leq \gamma.$$

By Szemerédi's Regularity Lemma / Theorem 50, there exists $M \in \mathbb{N}$ such that every graph of order at least m has an ε-regular partition $V_0 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} V_k$ with $m \leq k \leq M$.

Now, let G be a graph of order $n \in \mathbb{N}$. If $n < m$, then as matchings are interval colorable and $\Delta(G) \leq n - 1$, Theorem 2 gives us $\theta_{\text{int}}(G) \leq (n - 1) + 1 \leq M$.

Thus, we may assume that $n \geq m$. By our choice of M, we know that G has an ε-regular partition $V_0 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} V_k$ with $m \leq k \leq M$. Let $l := |V_1|$. For the construction of G', we proceed as follows: For each pair (X_i, X_j), $1 \leq i < j \leq k$, that is ε-regular with density greater than 7ε, let $G_{i,j}$ be the interval colorable factor guaranteed by Lemma 14, and otherwise let $G_{i,j} := (\emptyset, \emptyset)$. Then, we let

$$G' := \bigcup_{1 \leq i < j \leq k} G_{i,j}.$$

To see that $\theta_{\text{int}}(G') \leq M$, let c be a k-edge coloring of K_k, $V(K_k) = [k]$, which exists due to Theorem 2, and let

$$G_s := \bigcup_{1 \leq i < j \leq k, c(ij) = s} G_{i,j}$$

for $s \in [k]$. As each G_s is the vertex-disjoint union of interval colorable graphs, all G_s are interval colorable. So, $\theta_{\text{int}}(G') \leq k \leq M$.

We will bound the number of edges that are not in G'. For that, let

- x_1 be the number of edges in non-ε-regular pairs,
- x_2 be the number of edges induced by V_i for $0 \leq i \leq k$,

- x_3 be the number of edges in ε-regular pairs with density at most 7ε,

- x_4 be the number of edges in ε-regular pairs with density greater than 7ε that are not in G'.

For x_1, note that at most εk^2 of the pairs (V_i, V_j), $1 \le i < j \le n$, are non-ε-regular with there being at most $(n/k)^2$ in each such pair. Furthermore, $\|V_0, V_1 \cup \cdots \cup V_k\|$ is at most $|V_0| \cdot (|V| - |V_0|)$. As $\varepsilon < 1/2$, the maximum of that last expression is attained for $|V_0| = \varepsilon n$, so

$$x_1 \le \varepsilon k^2 \cdot \left(\frac{n}{k}\right)^2 + \varepsilon n(n - \varepsilon n) \le 2\varepsilon n^2.$$

For x_2, note that $|V_0| \le \varepsilon n$ and $|V_i| \le n/k$ for $1 \le i \le k$. This gives us

$$x_2 \le \binom{\varepsilon n}{2} + k\binom{n/k}{2} \le \frac{(\varepsilon n)^2}{2} + \frac{n^2}{2k}.$$

For x_3, a density of at most 7ε implies that there are at most $7\varepsilon(n/k)^2$ edges in that pair, so

$$x_3 \le \binom{k}{2} 7\varepsilon(n/k)^2 \le \frac{7}{2}\varepsilon n^2.$$

Lastly, for x_4, note that for a pair with parts of size l each and with density $d > 7\varepsilon$, at most

$$d \cdot l^2 - (d - 7\varepsilon) \cdot (l(1-\varepsilon))^2 \le 7\varepsilon^3 l^2 + 2d\varepsilon l^2 + 7\varepsilon l^2 \le 10\varepsilon \left(\frac{n}{k}\right)^2$$

edges are not in G'. So,

$$x_4 \le \binom{k}{2} 10\varepsilon(n/k)^2 \le 5\varepsilon n^2.$$

Therefore, the number of edges in $E(G) \setminus E(G')$ is

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &\le \left(2\varepsilon + \frac{\varepsilon^2}{2} + \frac{1}{2k} + \frac{7}{2}\varepsilon + 5\varepsilon\right) n^2 \\ &\le \left(\frac{1}{2m} + \frac{21+\varepsilon}{2} \cdot \varepsilon\right) n^2 \\ &\le \gamma n^2. \end{aligned}$$

This concludes the proof. □

While Theorem 51 doesn't give an immediate bound on the interval thickness of a graph itself, it allows us to focus our attention on graphs with only a fraction of the total number of possible edges. This situation can be dealt with Theorem 35.

Corollary 9. *For every $\varepsilon > 0$, there exists a sufficiently large $c \in \mathbb{N}$ such that $\theta_{\text{int}}(G) \le \varepsilon |G| + c$ for all graphs G.*

Proof. Set $\gamma \coloneqq 2\varepsilon^2$. By Theorem 51, there exists $M \in \mathbb{N}$ such that for every graph G there exists $G' \subseteq G$ with $\|G\| - \|G'\| \le \gamma |G|^2$ and $\theta_{\text{int}}(G') \le M$. Let $G'' = G - E(G') \subseteq G$, i.e. $\|G''\| \le \gamma |G|^2$. By Theorem 35,

$$\theta_{\text{int}}(G'') \le \sqrt{\frac{\gamma |G|^2}{2}} = \varepsilon |G|.$$

So, if we set $c \coloneqq M$, we get

$$\theta_{\text{int}}(G) \le \theta_{\text{int}}(G') + \theta_{\text{int}}(G'') < \varepsilon |G| + c.$$

This concludes the proof. $\square$

Of course, we may also "drop the constant" by only allowing large enough n:

Corollary 10. *For every $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for every $n > n_0$ and every n-vertex graph G $\theta_{\text{int}}(G) \leq \varepsilon n$.*

Remark 8. *Unlike the bounds given in [7] for example, our bound doesn't yield a polynomial algorithm as our proof heavily relies on Szemerédi's Regularity Lemma / Theorem 50: Although there are constructive versions of the Regularity Lemma given in [2] and [23], their running times have horrible dependency on ε and m. In the latter for example, $\mathcal{O}(\varepsilon^{-45})$ iterations are needed. Still, the fact that for general graphs a sublinear growing upper bound in the number of vertices was established shows that further improvements can be done.*

ON INTERVAL EDGE-COLORINGS OF BIPARTITE GRAPHS

S. V. Sevastianov (translated by M. Axenovich and M. Zheng)

1. Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E. For $a, b \in \{0, 1, 2, \ldots\}$, $a < b$, we denote by $(a, b]$ the interval $\{a + 1, \ldots, b\}$. A function $\varphi\colon E \to (0, t]$ is called an *edge-coloring* of G in t colors, where $\varphi(e)$ is called the color of an edge e, for $e \in E$. An edge-coloring of G is *proper* if any two adjacent edges have distinct colors. For $x \in V$, let $\varphi(x) = \{\varphi(xy) : \ xy \in E\}$. A proper edge-coloring $\varphi\colon E \to (0, t]$ of G is an *interval coloring* in t colors if $\varphi^{-1}(1) \neq \emptyset$, $\varphi^{-1}(t) \neq \emptyset$, and the set $\varphi(x)$ is an interval of integers for any vertex x.

Note that for any interval coloring $\varphi\colon E \to (0, t]$ of a graph (V, E) in t colors, there is a symmetric interval coloring $\varphi'\colon E \to (0, t]$, where $\varphi'(e) = t + 1 - \varphi(e)$ for any edge e. The notion of interval colorings is introduced in [1].

For arbitrary graph G, it was shown in [1] that the problem of interval colorability is $\mathcal{NP}$-complete. In the case of bipartite graphs, Kamalian [3] identified two subclasses of interval-colorable graphs: complete bipartite graphs and trees. Moreover, any regular bipartite graph is interval colorable. That follows from the fact that any bipartite graph has a proper edge-coloring using colors $\{1, \ldots, \Delta\}$, where Δ is the maximum degree of the graph, see [5, ca. 388].

In each of these three cases, the interval coloring would be found in polynomial time. It remains to determine whether any bipartite graph is interval colorable. An example giving a negative answer to this question is given in Figure 1. In this paper we show that the problem of determining whether a bipartite graph is interval colorable is $\mathcal{NP}$-complete.

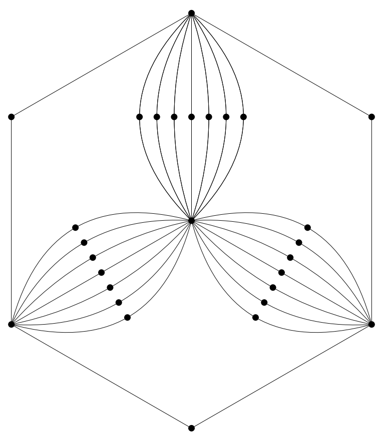

Figure 1

In [3] it was shown that trees and complete bipartite graphs satisfy the following property $(*)$: Let $w(G)$ and $W(G)$ be the minimum and maximum number of colors in an interval coloring of G. Then for any $t \in [w(G), W(G)]$, G is interval colorable in t colors.

It is easy to see that any regular interval colorable graph satisfies $(*)$. Indeed, if G is interval colorable in t colors, $(t > \Delta(G) = w(G))$, then by decreasing for each edge of color t its color by $\Delta(G)$, we shall get an interval coloring in $t-1$ colors.

It is natural to ask whether any interval colorable bipartite graph satisfies $(*)$. At the end of the paper we give an example giving a negative answer to this question.

2. The $\mathcal{NP}$-Completeness of ICBG

Interval Coloring of Bipartite Graphs (ICBG)

Instance: Simple bipartite graph $G = (V_1, V_2; E)$ and $t \in \mathbb{N}$.
Question: Is there an interval coloring of G in t colors?

We shall prove the $\mathcal{NP}$-completeness of this problem via pseudo-polynomial reduction of ICBG to a scheduling problem described below that is $\mathcal{NP}$-complete [2, pp. 102–104]. The definition of length$[H]$ and max$[H]$ is given in [2, pp. 92–95]. Roughly speaking, these functions correspond to the length of the input and the maximum of the absolute value of the input for a specific problem H, respectively.

Translator's remark

Note that ICBG is even strongly $\mathcal{NP}$-complete. Thus, the pseudo-polynomial reduction would indeed imply that ICBG is $\mathcal{NP}$-hard. The proof of the strong $\mathcal{NP}$-completeness of ICBG is given in Theorem 4.5 of [2, pp. 102–103].

Sequencing Within Intervals (SWI) [2, p. 70]

Instance: Finite set of tasks $N = \{1, 2, \dots, n\}$ and for each $i \in N$, $r_i, d_i, l_i \in \mathbb{Z}_+$ are the release time, deadline and length of i, respectively.
Question: Is there a schedule for N, i.e. a function $\sigma \colon N \to \mathbb{Z}_+$ such that $\forall\, i \in N$
(1) $\sigma(i) \geq r_i$,
(2) $\sigma(i) + l_i \leq d_i$,
(3) if $j \in N \setminus \{i\}$, then either $\sigma(j) + l_j \leq \sigma(i)$ or $\sigma(j) \geq \sigma(i) + l_i$?

Let a set of intervals of integers be given as $\{I_i \colon i = 1, \dots, m\}, I_i = (r_i, d_i] \subset \mathbb{Z}_+$. We shall construct a corresponding bipartite graph $G_{m,\{I_i\}}$, see Figure 2, where $l = m + 6; M = D + 2m + 10; D = D' + \delta(D'); D' = \max_i d_i$;

$$\delta(k) := \begin{cases} 0, & k \text{ even}, \\ 1, & k \text{ odd}. \end{cases}$$

Translator's remark

First, note that $\{I_i\}$ is a shorthand notation for $\{I_i \colon i = 1, \dots, m\}$. Also, observe that we can set D' to any value greater than or equal to $\max_i d_i$ and that it doesn't have to be $\max_i d_i$. This will be important in the third section. Lastly, we w.l.o.g. can assume that $r_i \leq d_i$ for all $i \in [m]$. Otherwise, one can easily check that the instance has no such schedule.

The vertices of one part of $G_{m,\{I_i\}}$ are labeled A with subscripts and superscripts, the vertices of the other part are labeled B with subscripts and superscripts; π_i is

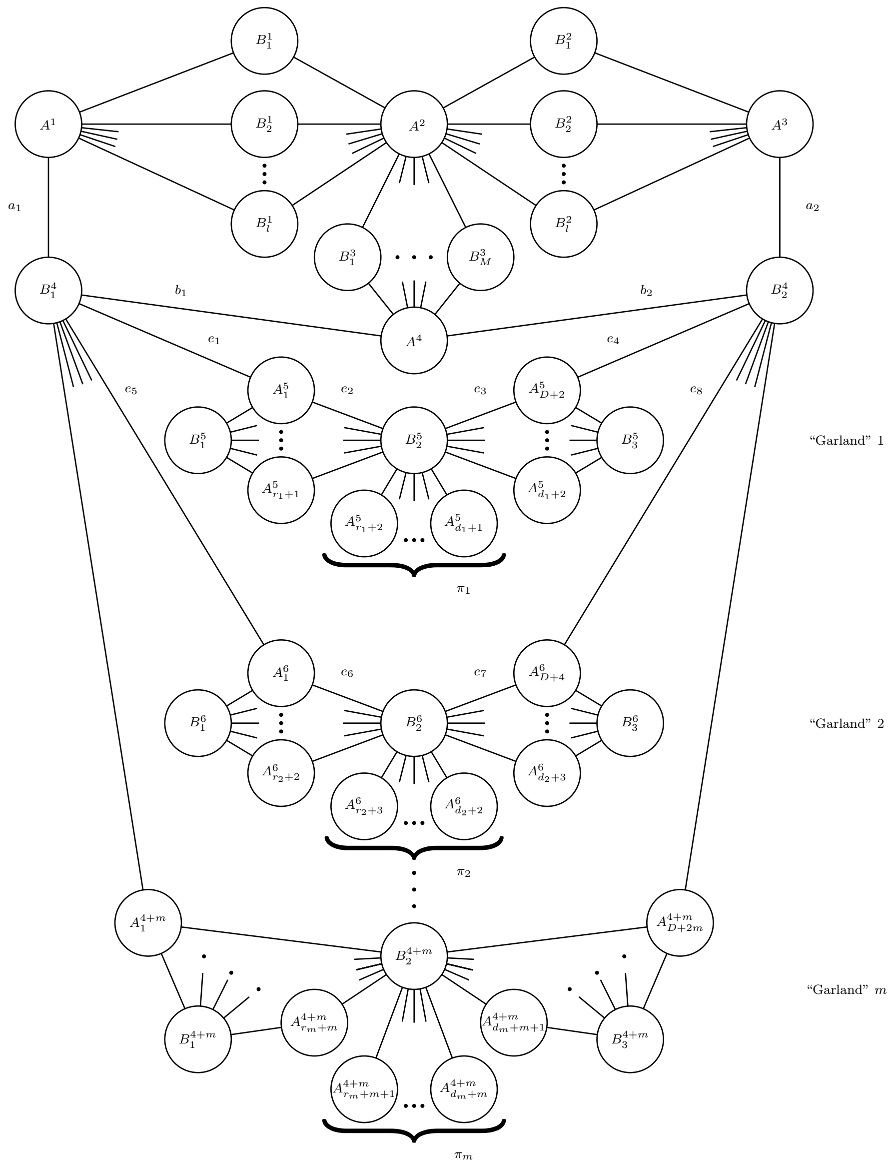

FIGURE 2. Graph $G_{m,\{I_i\}}$

a set of edges $\left\{B_2^{4+i}A_j^{4+i} \colon j = r_i + 1 + i, \dots, d_i + i\right\}$, we call this set π_i an *outer bundle.*

Lemma 1. *$G_{m,\{I_i\}}$ is interval colorable. Moreover, for any interval coloring φ the following holds:*

a) the number of colors is $T = D + 4m + 22 = M + 2l$;

b) edges $a_1 = A^1B_1^4$ and $a_2 = A^3B_2^4$ have colors $\varphi(a_1) = c_0, \varphi(a_2) = T + 1 - c_0$ or $\varphi(a_1) = T + 1 - c_0, \varphi(a_2) = c_0$, where $c_0 = m + 8$;

c) $\varphi(\pi_i)$ *is an interval for each* i*; moreover, if* $\varphi(a_1) = c_0$*, then* $\varphi(\pi_i) = c_1 + (r_i, d_i]$ *for any* $i = 1, \dots, m$*, where* $c_1 = 2m + 11$*.*

Proof. Let φ be an arbitrary interval coloring of $G_{m,\{I_i\}}$. Let $\varphi_1 \coloneqq \varphi(A^1) \cup \varphi(A^3) \cup \varphi(A^4)$, $\nu_1 \coloneqq \min \varphi_1$, $\mu_1 \coloneqq \max \varphi_1$. Let $\nu_2 = \min \varphi(A^2), \mu_2 = \max \varphi(A^2)$. Since $\mu_2 - \nu_2 = M + 2l - 1, \nu_1 \leq \nu_2 + 1, \mu_1 \geq \mu_2 - 1$, then

$$\mu_1 - \nu_1 \geq M + 2l - 3. \tag{1}$$

Translator's remark

Note that $\mu_2 - \nu_2 = M + 2l - 1$ follows from $\deg(A^2) = M + 2l$. Also, as some edge $A^2 B_i^j$ is colored ν_2, the other edge incident to B_i^j must have a color of size at most $\nu_2 + 1$, so – as the minimum color of φ_1 – we have $\nu_2 + 1 \geq \nu_1$. One can argue similarly for $\mu_1 \geq \mu_2 - 1$.

From this, it is clear that ν_1 and μ_1 could not belong to the same set $\varphi(A^i)$, $i \in \{1, 3, 4\}$. If $\nu_1 \in \varphi(A^4), \mu_1 \in \varphi(A^3)$, then because of (1) there should be a gap of at least

$$(\mu_1 - \nu_1 + 1) - \left|\varphi(A^4)\right| - \left|\varphi(A^3)\right| \geq (M + 2l - 2) - (M + 2) - (l + 1) = l - 5 = m + 1$$

colors between $\varphi(A^3)$ and $\varphi(A^4)$. This can not happen since $\varphi(A^4) \cap \varphi(B_2^4) \neq \emptyset, \varphi(A^3) \cap \varphi(B_2^4) \neq \emptyset, \left|\varphi(B_2^4)\right| = m + 2$, from which it follows that the gap between $\varphi(A^4)$ and $\varphi(A^3)$ is at most m. Thus, either

$$\nu_1 \in \varphi(A^1) \text{ and } \mu_1 \in \varphi(A^3) \text{ or } \nu_1 \in \varphi(A^3) \text{ and } \mu_1 \in \varphi(A^1). \tag{2}$$

Translator's remark

ν_1 and μ_1 could not belong to the same set $\varphi(A^i)$, $i \in \{1, 3, 4\}$, as otherwise $\max \varphi(A^i) - \min \varphi(A^i) + 1 = \deg(A^i) \leq M + 2 < M + 2l - 3$ would contradict (1).
Concerning the gap between $\varphi(A^3)$ and $\varphi(A^4)$, note that by definition $\varphi(A^3) \cup \varphi(A^4) \subset [\nu_1, \mu_1]$, so the gap is given by $\left|[\nu_1, \mu_1] \setminus (\varphi(A^3) \cup \varphi(A^4))\right|$.
Lastly, the case where $\mu_1 \in \varphi(A^3), \nu_1(A^4)$ or $\nu_1 \in \varphi(A^4), \mu_1 \in \varphi(A^1)$ can be similarly dealt with, leaving us only with the possibilities in (2).

Let's assume w.l.o.g. that the first statement of (2) holds. From (2), we have $\varphi(a_1) - \nu_1 \leq l, \mu_1 - \varphi(a_2) \leq l$, and from (1) we see

$$\varphi(a_2) - \varphi(a_1) \geq \mu_1 - l - \nu_1 - l \geq M - 3. \tag{3}$$

Translator's remark

Note that $\deg(A^1) = \deg(A^3) = l + 1$.

Consider a path (e_1, e_2, e_3, e_4) connecting the edges a_1 and a_2 (see Figure 2). Since φ is interval, we have

$$\begin{aligned}
|\varphi(a_1) - \varphi(a_2)| &\leq |\varphi(a_1) - \varphi(e_1)| + |\varphi(e_1) - \varphi(e_2)| + |\varphi(e_2) - \varphi(e_3)| \\
&\quad + |\varphi(e_3) - \varphi(e_4)| + |\varphi(e_4) - \varphi(a_1)| \\
&\leq (m + 1) + 2 + (D + 1) + 2 + (m + 1) \\
&= D + 2m + 7 \\
&= M - 3. \qquad (4)
\end{aligned}$$

Translator's remark

Note that the second inequality is merely the application of the fact that for a vertex v two colors in $\varphi(v)$ can at most differ by $\deg(v) - 1$.

From (3) and (4), it follows that first, $\varphi(a_2) - \varphi(a_1) = M - 3$, second, that all intermediate inequalities in (1), (3), and (4) hold as equalities. Thus,

$$\begin{aligned}
\nu_2 &= 1, \\
\nu_1 &= 2, \\
\mu_2 &= M + 2l = D + 4m + 22 = T, \\
\mu_1 &= M + 2l - 1, \\
\varphi(a_1) &= l + 2 = m + 8, \\
\varphi(a_2) &= M + l - 1 = M + m + 5; \\
\varphi(e_1) &= \varphi(a_1) + m + 1 = 2m + 9, \\
\varphi(e_4) &= \varphi(a_2) - m - 1 = M + 4,
\end{aligned}$$

i.e. the color $\varphi(e_1)$ is maximal in the interval $\varphi(B_1^4)$, and $\varphi(e_4)$ is minimal in $\varphi(B_2^4)$;

$$\varphi(e_2) = \varphi(e_1) + 2 = 2m + 11, \varphi(e_3) = \varphi(e_4) - 2 = M + 2,$$

i.e. $\varphi(e_2)$ and $\varphi(e_3)$ are minimal and maximal in interval $\varphi(B_2^5)$;

Translator's remark

$\nu_2 = 1$ and $\mu_2 = M + 2l$ seem to follow by implicitly assuming that $\varphi(A^2) = [1, \deg(A^2)] = [1, T]$, which one can do by suitably "shifting the coloring". However, it will turn out that that assumption is in some sense valid, as the coloring will then only use colors from $[1, T]$, which also shows that the graph is only colorable using T colors.

$$\begin{aligned}
\varphi((A_1^5, B_1^5)) &= \varphi(e_2) - 1, \\
\varphi((A_{D+2}^5, B_3^5)) &= \varphi(e_3) + 1, \\
\varphi(B_1^5) &= \{2m + 10, \ldots, 2m + 10 + r_1\}, \\
\varphi(B_3^5) &= \{2m + 13 + d_1, \ldots, M + 3\}, \\
\varphi(\pi_1) &= \{2m + 12 + r_1, \ldots, 2m + 11 + d_1\} = c_1 + (r_1, d_1].
\end{aligned}$$

Translator's remark

$\varphi((A_1^5, B_1^5)) = \varphi(e_2) - 1$ follows from $\varphi(e_1)$ and $\varphi(e_2)$ being minimal and maximal in $\varphi(A_1^5)$, meaning $\varphi((A_1^5, B_1^5))$ is in the center of $\varphi(A_1^5)$. Similarly, by the same reasoning with $\varphi(e_3)$ and $\varphi(e_4)$ for $\varphi(A_{D+2}^5)$, $\varphi((A_{D+2}^5, B_3^5)) = \varphi(e_3) + 1$.
For $\varphi(B_1^5)$, note that $\varphi(e_2) - 1 = 2m + 10$ must be the minimal color in it due to $\varphi(e_2)$ being the minimal color in $\varphi(B_2^5)$. $\varphi(B_1^5) = \{2m + 10, \ldots, 2m + 10 + r_1\}$ then follows from the degree of B_1^5. Similarly, for $\varphi(B_3^5)$, note that $\varphi(e_3) + 1 = M + 3$ must be the maximal color in it due to $\varphi(e_3)$ being the maximal color in $\varphi(B_3^5)$, which implies $\varphi(B_3^5) = \{2m + 13 + d_1, \ldots, M + 3\}$ by B_3^5's degree.

For $\varphi(\pi_1) = \{2m+12+r_1, \dots, 2m+11+d_1\} = c_1 + (r_1, d_1]$, observe that

$$\begin{aligned}
\max_{i\in\{1,\dots,r_1+1\}} \varphi((A_i^5, B_2^5)) &\le \max_{i\in\{1,\dots,r_1+1\}} \varphi((B_1^5, A_i^5)) + 1\\
&\le \max \varphi(B_1^5) + 1\\
&= 2m + 11 + r_1;\\
\min_{i\in\{d_1+2,\dots,D+2\}} \varphi((B_2^5, A_i^5)) &\ge \min_{i\in\{d_1+2,\dots,D+2\}} \varphi((A_i^5, B_3^5)) - 1\\
&\ge \min \varphi(B_3^5) - 1\\
&= 2m + 12 + d_1.
\end{aligned}$$

This leaves at least $(2m+11+d_1) - (2m+11+r_1) = d_1 - r_1 = |\pi_1|$ colors in $\varphi(B_2^5)$ not covered by the edges considered above. So, $\varphi(\pi_1) = \{2m+12+r_1, \dots, 2m+11+d_1\}$ and the inequalities above are tight.
Note that these assignments of colors can indeed be extended to a full interval coloring of the garland by setting

$$\forall\, i \in \{1,\dots,r_1+1\} : \varphi((A_i^5, B_2^5)) = \varphi((B_1^5, A_i^5)) + 1$$
$$\forall\, i \in \{d_1+2,\dots,D+2\} : \varphi((B_2^5, A_i^5)) = \varphi((A_i^5, B_3^5)) - 1.$$

Considering the second "garland", that hangs on B_1^4, B_2^4 and contains the bundle π_2, note that $\varphi(e_5) = \varphi(e_1) - 1, \varphi(e_8) = \varphi(e_4) + 1, \varphi(e_8) - \varphi(e_5) = M + 4 - 2m - 9 + 2 = D + 2m + 10 - 2m - 3 = D + 7$, from which we have $\varphi(e_7) - \varphi(e_6) = D + 3$, i.e. again $\varphi(e_6)$ and $\varphi(e_7)$ are the minimum and maximum colors in $\varphi(B_2^6)$, from which we analogously obtain $\varphi(\pi_2) = c_1 + (r_2, d_2]$, and so on...

Translator's remark

For this paragraph, it is important to keep in mind that $\varphi(B_1^4) = [\varphi(a_1), \varphi(e_1)]$ and $\varphi(B_2^4) = [\varphi(e_4), \varphi(a_2)]$ by the minimality and maximality of the respective colors for B_1^4 and B_2^4. So, for the second "garland", $\varphi(e_5)$ can be at most $\varphi(e_1) - 1$ and $\varphi(e_8)$ at least $\varphi(e_4) + 1$. To choose the colors as such turns out to be necessary: By the same argument using the path (e_5, e_6, e_7, e_8), we get

$$|\varphi(e_8) - \varphi(e_4)| \le (\deg(A_1^6) - 1) + (\deg(B_2^6) - 1) + (\deg(A_{D+4}^6) - 1) = D + 7,$$

which is $(\varphi(e_4) + 1) - (\varphi(e_1) - 1)$.
In general, note that $\deg(A_1^{4+i}) = \deg(A_{D+2i}^{4+i}) = 3$ and $\deg(B_2^{4+i}) = D + 2i$ for $i \in [m]$. So the path argument will give for all $i \in [m]$

$$\begin{aligned}
&\left|\varphi((B_2^4, A_{D+2i}^{4+i})) - \varphi((B_1^4, A_1^{4+i}))\right|\\
&\le (\deg(A_1^{4+i}) - 1) + (\deg(B_2^{4+i}) - 1) + (\deg(A_{D+2i}^{4+i}) - 1)\\
&= D + 2i + 3,
\end{aligned}$$

which is tight for $i \in \{1, 2\}$. Inductively, it is therefore clear that $\varphi((B_1^4, A_1^{4+i})) = \varphi(e_1) - (i-1)$ and $\varphi((B_2^4, A_{D+2i}^{4+i})) = \varphi(e_4) + (i-1)$ and that for all $i \in [m]$ the inequality is actually an equality. This also agrees with the intervals $\varphi(B_1^4)$ and $\varphi(B_2^4)$.

By generalizing our analysis of the first "garland" to all of the "garlands" and using $\varphi(e_1) = 2m + 9$, $\varphi(e_4) = M + 4$, it follows for $i \in [m]$

$$\begin{aligned}
\varphi((A_1^{4+i}, B_1^{4+i})) &= \varphi((B_1^4, A_1^{4+i})) + 1 = 2m + 11 - i,\\
\varphi((A_{D+2i}^{4+i}, B_3^{4+i})) &= \varphi((B_2^4, A_{D+2i}^{4+i})) - 1 = M + 2 + i,\\
\varphi(B_1^5) &= \{2m + 11 - i, \ldots, 2m + 10 + r_i\},\\
\varphi(B_3^5) &= \{2m + 13 + d_i, \ldots, M + 2 + i\},\\
\varphi(\pi_1) &= \{2m + 12 + r_i, \ldots, 2m + 11 + d_i\} = c_1 + (r_i, d_i].
\end{aligned}$$

To look at a "generic garland", we refer to Figure 3, which wasn't originally in the paper.

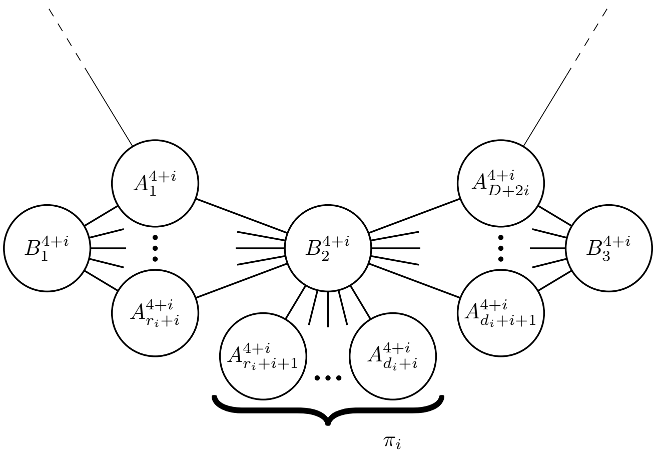

FIGURE 3. Anatomy of the i-th garland

From the proven properties, it is not difficult to reconstruct the interval coloring of the whole graph. Since we basically gave an algorithm of coloring "garlands", it remains to explain how we color the central part of the graph. In Figure 4, it is shown how the colors from $\varphi(A^1), \varphi(A^4), \varphi(A^3)$ match the colors in $\varphi(A^2)$ at vertices B_j^i, $i \in \{1, 2, 3\}$.

A diagonal line connects two neighboring colors (one from $\varphi(A^2)$, another from one of the intervals $\varphi(A^1), \varphi(A^3), \varphi(A^4)$), that appear at some vertex B_j^i, $i \in \{1, 2, 3\}$. In addition, pairing up consecutive colors in $\{\varphi(b_1) + 1, \ldots, \varphi(b_2) - 1\} = \{m + 10, \ldots, M + m + 3\} \subset \varphi(A^4)$ (see Figure 4) with the same colors from $\varphi(A^2)$ at B_j^3 is guaranteed to work due to the parity of the parameter D.

Translator's remark

Note that the dashed vertical lines separate the individual colors of the intervals, with the intervals being positioned such that equal colors are horizontally aligned. Furthermore, note that the supposed "gaps" in $\varphi(A^4)$ for the colors $\varphi(b_1)$ and $\varphi(b_2)$ are due to A_4 being incident to b_1 and b_2.

$\square$

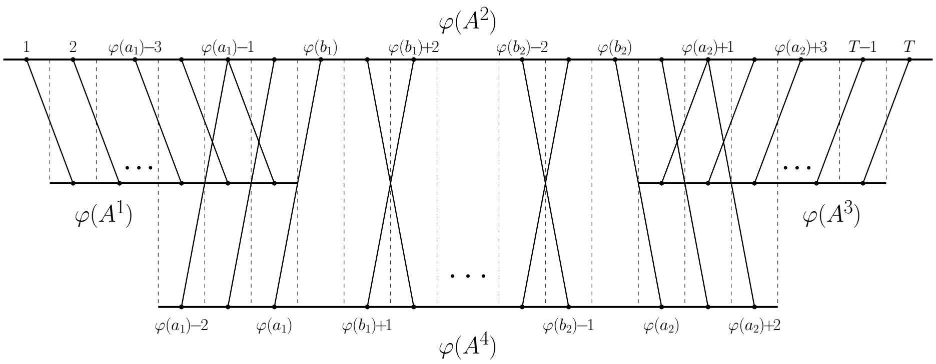

FIGURE 4

Theorem 2. *The problem* ICBG *is* $\mathcal{NP}$*-complete.*

Translator's remark

The following proof technically only shows that ICBG is $\mathcal{NP}$-hard. But whether an edge coloring is also an interval coloring can be checked in polynomial time, so it is immediate that ICBG is in $\mathcal{NP}$.

Proof. Let us reduce the problem of SWI to ICBG. Let H be a specific instance of SWI with initial conditions $\{r_i, d_i, l_i : i = 1, \ldots, n\} \subset \mathbb{Z}_+, \max[H] = D', \text{length}[H] \leq M \log_2 D'$. Let us construct a bipartite graph $G(H)$ as shown in Figure 5.

We can assume that the following conditions are satisfied:

$$r_i + l_i \leq d_i, i = 1, \ldots, n, \tag{5}$$

$$L \doteq \sum_{i=1}^{n} l_i \leq D'. \tag{6}$$

If some condition above is not satisfied, (that could be checked in linear of length$[H]$ time), then obviously there is no feasible schedule in problem H.

Translator's remark

Concerning $\max[H] = D'$, it is important to note that for that condition (5) must hold which as stated we can easily assume. The exact bound on length$[H]$ isn't very immediate, although for our analysis only the asymptotic behaviour is important: Each number of the initial conditions can be encoded in $\mathcal{O}(\log_2 D')$ binary digits, of which there are $3n \in \mathcal{O}(M)$ many, so length$[H] \in \mathcal{O}(M \log_2 D')$.

The graph $G(H)$ consists of the following parts:

- a graph $G^* = G_{2n,\{I_i, 2+I_i : i=1,\ldots,n\}}$, where $I_i = (r_i, d_i]$,
- a graph $G^{**} = G_{2,\{(0,D'], 4+(0,D']\}}$,
- vertices $\{v_i : i = 1, \ldots, n\}$ with bundles of edges $\pi(v_i)$, $|\pi(v_i)| = l_i$,
- additional vertices $\left\{v_j^i : j = 1, \ldots, d_i - r_i - l_i; i = 1, \ldots, n\right\}$,
- additional vertices $\left\{\overline{v}_j^i : j = 1, 2, 3; i = L + 1, \ldots, D'\right\}$.

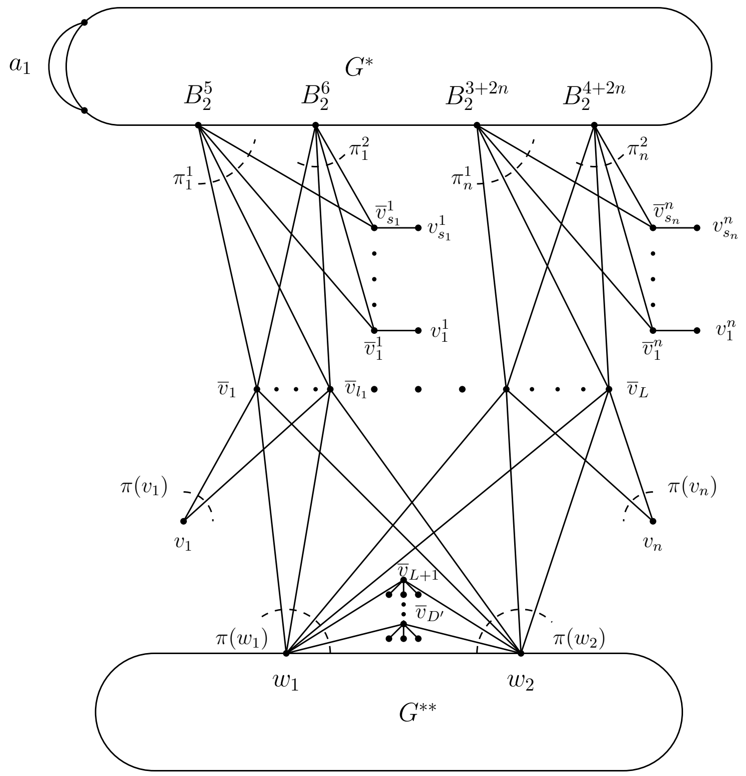

FIGURE 5. Graph $G(H)$

We allow for some additional vertices not to be included if inequalities in (5) or (6) for corresponding i hold as equalities. In G^*, the outer bundles π_i^1 and π_i^2 correspond to intervals I_i and $2+I_i$. In G^{**} there are two outer bundles incident to vertices w_1 and w_2. Pendant vertices of the bundles $\{\pi_i^1, \pi_i^2, \pi(v_i), \pi(w_1), \pi(w_2)\}$ are connected as follows: In vertices $\overline{v}_i, i = 1, \ldots, L$, the edges of the following bundles are connected $\{\pi_i^1, \pi_i^2, \pi(v_i), \pi(w_1), \pi(w_2)\}$.

If $s_i \doteq d_i - r_i - l_i > 0$, then the remaining s_i edges of the bundle π_i^1 and s_i edges from π_i^2 are pairwise connected at vertices $\overline{v}_j^i, j = 1, \ldots, s_i$, adjacent to additional vertices v_j^i. The degree of each vertex $\overline{v}_j^i$ is 3.

If $D' - L > 0$, then the remaining $D' - L$ edges of the bundle $\pi(w_1)$ and $D' - L$ edges of the bundle $\pi(w_2)$ are pairwise connected at $\overline{v}_i, i = L+1, \ldots, D'$, adjacent also to additional vertices $\{\overline{v}_j^i : j = 1, 2, 3\}$.

The degree of each $\overline{v}_i, i = 1, \ldots, D'$ is 5; all additional vertices are pendant.

Next we shall show that a feasible schedule of H exists if and only if there is an interval coloring of $G(H)$. Let φ be an interval coloring of $G(H)$. Then by Lemma 1, G^* is colored in $T = D + 8n + 24$ colors from $(0, T]$.

Translator's remark

It might seem at first that T should be $D + 8n + 22$. However, looking at G^*, its maximal deadline is $D' + 2$ due to the shifting, so its "D" is $D + 2$ as the parity of D' and $D' + 2$ are the same. Thus, the lemma implies that one needs exactly $T = (D + 2) + 4(2n) + 22 = D + 8n + 24$ colors.
It is also important that, although the pendant edges of G^* are connected in $G(H)$, restricting the coloring onto G^*'s edges lets us obtain an interval coloring of G^* as "reverting" the connections of originally pendant edges preserves the interval colorability. So, Lemma 1 can be applied. However, to conclude that G^* is colored in T colors from $(0, T]$ seems again to be an assumption "up to shifts" of the coloring. Again, however, as shown below, the colors rightfully turn out to be in $(0, T]$.

We shall show that all other edges of $G(H)$ are colored in colors from this interval $(0, T]$: Assume that $\varphi(a_1) = 2n+8$ (otherwise consider a symmetric coloring). Then by Lemma 1,

$$\varphi(\pi_i^1) = 4n + 11 + (r_i, d_i], \qquad \varphi(\pi_i^2) = 4n + 13 + (r_i, d_i].$$

From the lemma, we also know that $\varphi(\pi(w_1))$ and $\varphi(\pi(w_2))$ are two intervals, one obtained from another by shifting by 4. Let w_1 denote the vertex corresponding to the left interval, w_2 to the right interval. Pairwise adjacency of the edges of the bundles $\pi(w_1), \pi(w_2)$ appearing at $\overline{v}_j$, establishes a bijection between $\varphi(\pi(w_1))$ and $\varphi(\pi(w_2))$. Because $|\varphi((\overline{v}_j, w_1)) - \varphi((\overline{v}_j, w_2))| \le 4$, $j = 1, \dots, D'$, it is not difficult to see that for each j these are equalities, i.e. in the set of 5 colors $\varphi(\overline{v}_j)$ the colors $\varphi((\overline{v}_j, w_1)), \varphi((\overline{v}_j, w_2))$ are the minimum and maximum elements respectively.

Translator's remark

We will elaborate on why those have to be equalities: The smallest color m_1, m_2 of $\varphi(\pi(w_1))$, $\varphi(\pi(w_2))$ differ by 4. So, if m_1 isn't paired up with m_2, the difference of the colors would be greater than 4, contradicting the inequalities. Hence, m_1 is paired up with m_2, giving us equality for the corresponding edges. The remaining, "unpaired" colors $\varphi(\pi(w_1)) \setminus \{m_1\}$ and $\varphi(\pi(w_2)) \setminus \{m_2\}$ are again intervals with $\varphi(\pi(w_2)) \setminus \{m_2\} = \varphi(\pi(w_1)) \setminus \{m_1\} + 4$. So, the claim follows inductively.

From this, we have $\left|\varphi((\overline{v}_j, B_2^{3+2i})) - \varphi((\overline{v}_j, B_2^{4+2i}))\right| \le 2$, and similar arguments give

$$\varphi((\overline{v}_j, B_2^{4+2i})) = \varphi((\overline{v}_j, B_2^{3+2i})) + 2.$$

Therefore, in each interval $\varphi(\overline{v}_j)$, the color $\varphi((\overline{v}_j, v_i))$ is in the center of the interval. From that we have

$$\varphi(v_i) \subset \varphi(\pi_i^1) + 1 = 4n + 12 + (r_i, d_i], i = 1, \dots, n,$$

and the intervals $\{\varphi(v_i)\colon i = 1, \dots, n\}$ do not intersect since the corresponding intervals of the edges of the bundle $\pi(w_1)$,

$$\varphi\left(\left\{(\overline{v}_j, w_1)\colon j = \sum_{k=1}^{i-1} l_k + 1, \dots, \sum_{k=1}^{i} l_k\right\}\right) = \varphi(v_i) - 2, i = 1, \dots, n,$$

do not intersect. (By the way, it allows us to establish that $\varphi(G^{**}) = (4n - 5, D + 4n + 29] \subset (0, D + 8n + 24] = \varphi(G^*)$ for any $n > 1$.)

Translator's remark

Concerning $\varphi(G^{**})$, we first note that w.l.o.g. we may assume that $\min_i r_i = 0$. Otherwise, we may consider the equivalent instance of SWI with $\{r'_i, d'_i, l_i : i = 1, \dots, n\}$ where $r'_j = r_j - \min_i r_i$ and $d'_j = d_j - \min_i r_i$. Note that this transformation and re-transformation can be done in linear time. Now, it is important to note that the minimum color of any of the bundles $\{\pi_i^1, \pi_i^2\}$ is $\min_i(\min \varphi(\pi_i^1)) = 4n + 12 + \min_i r_i$, while the maximum color is $\max_i(\max \varphi(\pi_i^2)) = 4n + 13 + \max_i d_i = 4n + 13 + D'$. Thus, the minimum possible color assignable to an edge in $\pi(w_1) \cup \pi(w_2)$ is $4n + 11 + \min_i r_i$ and the maximum possible color assignable is $4n + 14 + D'$ by our previous observations. So, the difference of the maximum and minimum color in $\varphi(\pi(w_1) \cup \pi(w_2))$ is at most $(4n + 14 + D') - (4n + 11 + \min_i r_i) = D' + 3 - \min_i r_i = D' + 3$.
However, as $\varphi(\pi(w_1))$ and $\varphi(\pi(w_2))$ are intervals of length D' where one can be shifted by 4 to obtain the other interval, the difference of the maximum and minimum color in $\varphi(\pi(w_1) \cup \pi(w_2))$ is exactly $D' + 3$. Thus, $\varphi(\pi(w_1)) = (4n + 10, 4n + 10 + D']$ and $\varphi(\pi(w_2)) = (4n + 14, 4n + 14 + D']$.
Now, focusing only on G^{**}, note that "its m" is 2, "its D'" is $D' + 4$, "its D" is $D + 4$, "its M" is $D + 18$. Hence, Lemma 1 implies that any interval coloring ψ of it uses $D + 34$ colors and either $\psi(\pi(w_1)) = (15, D' + 15]$ and $\psi(\pi(w_2)) = (19, D' + 19]$, or $\psi(\pi(w_1)) = (D + 35) - (19, D' + 19] = (\delta(D') + 15, D + 15]$ and $\psi(\pi(w_2)) = (D + 35) - (15, D' + 15] = (\delta(D') + 19, D + 19]$.
Comparing that with $\varphi(\pi(w_1))$ and $\varphi(\pi(w_2))$, we get

$$\varphi(G^{**}) \in \left\{(4n - 5, D + 4n + 29], (4n - 5 - \delta(D'), D' + 4n + 29]\right\},$$

both of which are subsets of $(0, T] = \varphi(G^*)$ for any $n > 1$.
Keep in mind that the colors of the additional edges $\varphi((\overline{v}_j^i, v_j^i))$ for $i \in [n], j \in [s_i]$ are always in the center of $\varphi(\overline{v}_j^i)$ and that for $i = L + 1, \dots, D', j = 1, 2, 3$

$$4n + 10 \leq \varphi((\overline{v}_i, w_1)) - 1 \leq \varphi((\overline{v}_i, \overline{v}_j^i)) \leq \varphi((\overline{v}_i, w_2)) + 1 \leq 4n + 15 + D'.$$

Therefore, $\varphi(G(H)) = (0, T]$.

From this, it is clear that assigning the i-th task to the interval $\varphi(v_i) - (4n+12)$ gives us a feasible schedule for H.

Translator's remark

In other words, $\sigma(i) := \min_i \varphi(v_i) - (4n+13)$ for $i \in [n]$ defines a feasible schedule.

Proving the other direction is not difficult since it is clear how given a schedule of H to get a coloring of $G(H)$. Since the number of edges of $G(H)$ is equal to

$$D \cdot (4n + 8) + D' + 8n^2 + 32n + 156 - 3L - \sum_{i=1}^{n}(d_i - r_i),$$

it does not exceed a polynomial of n and D' (and thus of length$[H]$ and max$[H]$).

Translator's remark

We may first count the number of edges in a "generic" $G_{m,\{I_i\}}$. For it, we can split the edges into

1. the edges incident to A_2,
2. the edges incident to A_1 and A_3,

3. the edges incident to A^4,
4. the edges "contained" in a "garland".

The edges of each "garland" can be further divided up into the ones incident to B_2^{4+i}, the two outer edges incident to B_1^4 and B_2^4 respectively, and the ones incident to B_1^{4+i} and B_3^{4+i}. From that, we get that the number of edges is

$$
\begin{aligned}
&M + 2l + 2(l+1) + M + 2 + \sum_{i=1}^{m}((D+2i) + (r_i + i) + (D + 2i - d_i - i) + 2) \\
&= 2M + 4l + 4 + \sum_{i=1}^{m}(2D + 4i + 2 - (d_i - r_i)) \\
&= 2M + 4l + 4 + 2Dm + 2m(m+1) + 2m - \sum_{i=1}^{m}(d_i - r_i) \\
&= 2D + 12m + 2Dm + 2m^2 + 48 - \sum_{i=1}^{m}(d_i - r_i).
\end{aligned}
$$

So, we get for the number of edges

$$
\begin{aligned}
|E(G^*)| &= 2(D+2) + 24n + 4(D+2)n + 8n^2 + 48 - 2\sum_{i=1}^{n}(d_i - r_i) \\
&= 2D + 32n + 4Dn + 8n^2 + 52 - 2\sum_{i=1}^{n}(d_i - r_i) \\
|E(G^{**})| &= 2(D+4) + 24 + 4(D+4) + 8 + 48 - 2D' \\
&= 6D + 104 - 2D'.
\end{aligned}
$$

For the remaining edges of $G(H)$, we can split them into
1. the edges $\pi(v_i)$ $(i \in [n])$,
2. the edges incident to v_j^i $(i \in [n], j = 1, \ldots, s_i)$,
3. the edges incident to v_j^i $(i = L+1, \ldots, D', j \in [3])$.

This gives us

$$
\sum_{i=1}^{n}(l_i + s_i) + 3(D' - L) = 3(D' - L) + \sum_{i=1}^{n}(d_i - r_i)
$$

remaining edges, giving us

$$
|E(G(H))| = D \cdot (4n+8) + D' + 8n^2 + 32n + 156 - 3L - \sum_{i=1}^{n}(d_i - r_i).
$$

Therefore, if there was an algorithm for ICBG that is polynomial on the number of edges of the graph, one would have gotten an in length$[H]$, max$[H]$ polynomial algorithm for SWI, for any instance H. This contradicts the strong $\mathcal{NP}$-completeness of SWI (assuming $\mathcal{P} \neq \mathcal{NP}$). □

3. A Bipartite Graph Without Property (∗)

Next we shall construct a bipartite graph G, not satisfying (∗). Let $G_0 = G_{m,\{I_i\}}$ (as defined in Lemma 1) with $m = 1, D = D' = 74, I_1 = (r_1, d_1] = (0, 1]$.

According to Lemma 1, G_0 is interval colorable in $T = 100$ colors, and the color of the outer bundle consisting of a single edge is 14 (or in the symmetric coloring 87). Take G as the vertex disjoint union of two copies G', G'' of G_0 and then identify the outer edge of G' with the outer edge of G'' oriented in opposite direction, i.e.

B_2^5 in one graph is identified with A_2^5 in another graph and vice-versa, see Figure 6. Then one can interval color G in 100 and 173 colors, but for any $t \in (100, 173)$, there is no interval coloring of G in t colors.

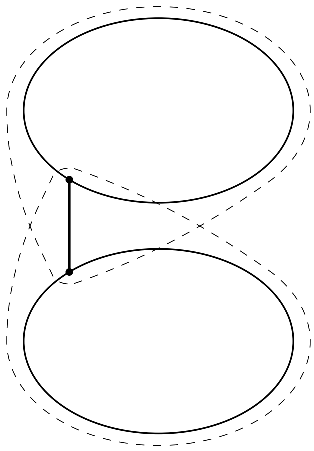

FIGURE 6. Graph G

4. CONCLUSION

The constructed graph in this paper and the main theorem show that properties of complete bipartite graphs and trees such as property $(*)$ found by Kamalian and the existence of a polynomial algorithm for interval coloring, do not hold for all bipartite graphs and thus are non-trivial properties for these graph classes.

Using this occasion, I would like to thank Ageev who noticed that the considered problem is related to the known problem 2-DIMENSIONAL CONSECUTIVE SETS, see [2, p. 230] under number [SR19]. It is equivalent to the interval colorability problem of an arbitrary hypergraph, whose $\mathcal{NP}$-completeness was shown in [4] long before Asratian and Kamalian introduced the concept of interval coloring a graph.

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Chapter 9

Concluding remarks

In this thesis, we have given a broad overview into the study of interval colorings. On our way, we stumbled upon various open problems and conjectures. In chapter 5, we investigated the interval colorability of particular classes of biregular graphs and, building on the paper by Sevastianov, showed in Theorem 49 that the spectrum of a graph can have arbitrarily many and arbitrarily large gaps, even if we restrict ourselves to planar bipartite graphs. Lastly, we have shown in Corollary 10 that the interval thickness grows sublinearly in the number of vertices which is an improvement over currently known upper bounds for dense graphs.

However, the upper bound given still leaves a lot of room for improvement and we hope that the work of this thesis may contribute to resolve the following conjecture attributed to Axenovich (see [16]).

CONJECTURE 6. *For every graph G we have $\theta_{\text{int}}(G) \leq 2$.*

There is strong reason to believe that the conjecture is true for the simple fact that in the more than 30 years of research into this topic, no graph with interval thickness greater than two is known. Analogous results are also true for similar properties.

LEMMA 15. *Every simple graph can be edge-decomposed into two Class 1 graphs.*

PROOF. The statement is trivial for Class 1 graphs. So, let $G = (V, E)$ be a Class 2 graph, $v \in V$ be a vertex of maximum degree and c be a proper edge coloring of G using $\Delta(G) + 1$ colors. Note that none of the color classes of c can be empty as otherwise G could be edge colored using only $\Delta(G)$ colors. Furthermore, as there are $\Delta(G) + 1$ color classes and only $\Delta(G)$ edges incident to v, there must be a color class C that contains no edge incident to v. Let $G_1 \coloneqq (V, E \setminus C)$ and $G_2 \coloneqq (V, C)$.

G_2 is a matching, so only one color is necessary for a proper edge coloring. Thus, G_2 is a Class 1 graph. Furthermore, G_1 must have the same maximum degree as G since v has degree $\Delta(G)$ in G_1 by construction. An edge coloring of G_1 using $\Delta(G)$ colors can be therefore easily derived from c. Thus, G_1 is also a Class 1 graph.

As G_1 and G_2 edge-decompose G, we are done. $\square$

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