The BOOK of Graph Theory II

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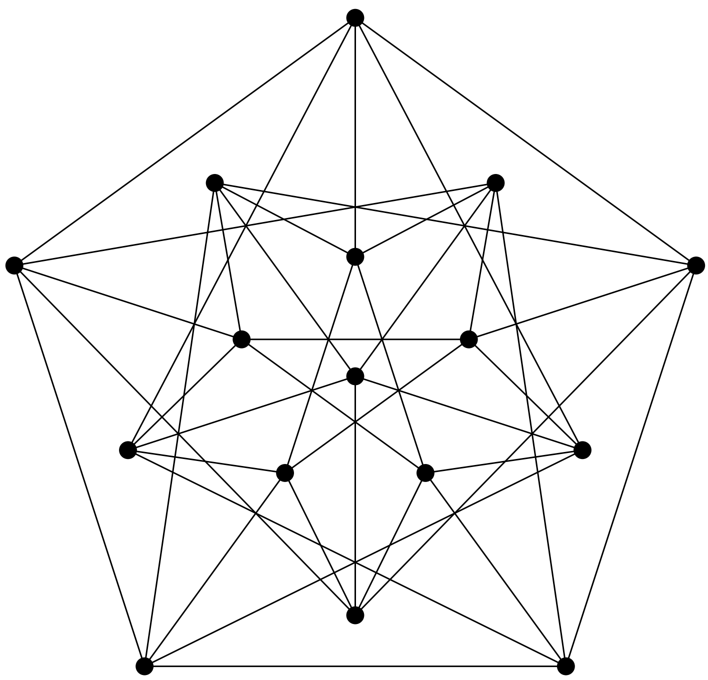
"If I have seen further than others, it is by standing on the shoulders of giants."

– Isaac Newton

This document is dedicated to the best solutions (or at least attempt thereof) to exercise problems of *Graph Theory II* at the University of Hamburg. Specifically, the document will discuss all problems that were on exercise sheets of the course in the winter semester 2022 - 2023. Whenever I refer to a theorem, lemma, etc. I refer to the corresponding theorem, lemma, ... in Diestel's *Graph Theory* where I am using the fifth edition. The solutions may serve as a substitute for the (disappointing) lack of model solutions to the (fairly) consistent problem sets of this course. I hope it will help students in the future, especially those struggling from a less wide graph theory background. It may *not* be used for cheating or copying. Good luck and *viel Erfolg!*



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Sheet 1

Exercise 1

For a graph $G = (V, E)$ and $|X| \subseteq V$ let $N_{\geq 2}(X)$ be the set of vertices outside X having at least two neighbours in X, i.e.

$$N_{\geq 2}(X) = \{v \in V \setminus X \colon\ |N(v) \cap X| \geq 2\}\,.$$

Suppose $|N_{\geq 2}(X)| \geq |X|$ for every independent subset $X \subseteq V$ of size at least 2.
Then G contains a matching covering all but at most one vertex.

Proof. Without loss of generality we may assume that G is non-empty. By the stronger version of Tutte's Theorem, it suffices to show that $\operatorname{def}(G) \leq 1$. As in Diestel, let $q(S)$ denote the number of odd components in $G - S$.

Case 1: $S = \emptyset$. We will show that $q(S) \leq 1$. For the sake of contradiction, assume that $q(S) \geq 2$, i.e. there are at least two (different) odd components $C^1, C^2 \subseteq G - S$. Let $x^1 \in V(C^1), x^2 \in V(C^2)$ be two vertices of the given components and consider $X = \{x^1, x^2\}$. X forms an independent set as they are in different components in $G - S$. Since $|N_{\geq 2}(X)| \geq |X| = 2$ by assumption, there must be some vertex $y \in V(G)$ which is adjacent to both x^1 and x^2 which means that they must actually be in the same connected component. $\lightning$

Case 2: $S \neq \emptyset$. Again, for the sake of contradiction, assume that $q(S) > |S|$. Let $C^1, \ldots, C^{q(S)}$ be the different components and let x^i be an arbitrary vertex from C^i ($i \in [q(S)]$). Consider $X = \{x^i \mid i \in [q(S)]\}$. By construction, X is an independent set of size $q(S)$. In particular, we get $|N_{\geq 2}(X)| \geq |X| > |S|$. However, $N_{\geq 2}(X) \subseteq S$. $\lightning$

This completes the proof. □

Alternative proof. According to the Gallai-Edmonds Structure Theorem[1], there is a vertex set S in G such that:

- (i) S is matchable to $\mathcal{C}_{G-S}$[2].
- (ii) Every component of $G - S$ is factor-critical.

Take out of every component in $\mathcal{C}_{G-S}$ a vertex and put them in a set X. Then $|X| = |\mathcal{C}_{G-S}|$ and X is independent. So, we have that $|N_{\geq 2}(X)| \geq |X|$. However, note that $N_{\geq 2}(X) \subseteq S$, meaning that

$$|S| \geq |N_{\geq 2}(X)| \geq |X| = |\mathcal{C}_{G-S}| \overset{(i)}{\geq} |S| \implies |S| = |\mathcal{C}_{G-S}|\,.$$

(i) in particular then implies that you can create a perfect matching by matching each vertex in S to one of some component in $\mathcal{C}_{G-S}$ and then utilizing the factor-criticality of each component. □

[1] Theorem 2.2.3 in Diestel.
[2] The set of components of $G - S$.

Exercise 2

A graph $G = (V, E)$ is *vertex transitive* if, for any two vertices $v, w \in V$, there is an automorphism φ of G with $\varphi(v) = w$. Then every connected, vertex-transitive graph with an even number of vertices contains a perfect matching.

Elementary Proof. Suppose for a contradiction that we don't have a perfect matching and pick an S according to Gallai-Edmonds Structure Theorem / Theorem 2.2.3. Since the graph is connected and has an even number of vertices, S must be non-empty.
Recall that every component of $G - S$ (denoted by $\mathcal{C}_{G-S}$) is odd due to (ii) in Theorem 2.2.3. Hence, by the choice of S as the set with the worst deficiency, the stronger version of Tutte's Theorem asserts that a maximum matching covers all but $|\mathcal{C}_{G-S}| - |S|$ vertices. Thus, every maximum matching induces a matching (covering S) between S and $\mathcal{C}_{G-S}$. Since every maximum matching needs to cover S, it follows that there is some $v \in S$ that is covered in all maximum matchings. Let M be a maximum matching and let u be a vertex uncovered by M. By the vertex transitivity there is an automorphism φ that brings u to v. Then the image of M under φ is a maximum matching avoiding v, a contradiction. $\square$

Proof using the full statement of the Gallai-Edmonds Structure Theorem. Note that an automorphism φ of G always preserves matchings in the sense that if M is a matching, then $\varphi(M) := \{\varphi(v)\varphi(w) \mid vw \in M\}$ is a matching of the same size.
Since G is transitive, we either have that

- for every $v \in V(G)$ there is a maximum matching avoiding v, or that
- for every $v \in V(G)$ every maximum matching covers v.

Indeed, if there is a maximum matching M avoiding some $v \in V$, then we can apply the automorphism φ of G with $\varphi(v) = w$ and $\varphi(M)$ would then be a maximum matching avoiding $\varphi(v) = w$.
In the latter case, a maximum matching would be perfect, thus we would done.
So, for the sake of contradiction, consider the first case. Let $D(G) \in V(G)$ be the set of vertices that are avoided by some maximum matching. According to the Gallai-Edmonds Structure Theorem / Theorem 2.2.3[3], S, the set of vertices $v \notin D(G)$ with $N(v) \cap D(G) \neq \emptyset$, is a set of maximal deficiency, i.e. $\operatorname{def}(G) = \max_{T \subseteq V} q(G - T) - |T| = q(G - S) - |S|$. As we are in the first case, $S = \emptyset$. However, for $S = \emptyset$, we have a deficiency of 0 as G is even and connected. This means that, contrary to our assumptions, there is a perfect matching (by the stronger version of Tutte's Theorem) which in particular covers every vertex of $V(G)$. $\square$

Bonus Question (Theorem 2.2.3 $\implies$ Gallai-Edmonds Structure Theorem)**.** Let G be a graph. Then $S \subseteq V(G)$ can be chosen in such a way that S is strongly matchable to the odd components in $G - S$, which we denote by $\mathcal{C}_{G-S}$ and which are all factor-critical, and all even components in $G - S$ have perfect matchings. Furthermore, S is unique.

[3] Theorem 2.2.3 is only a lean version of the full Gallai-Edmonds Structure Theorem.

Proof. As in the proof of Theorem 2.2.3, choose S' in such a way that maximum deficiency $q(G - S') - |S'|$ and is of maximal size. We have shown that $G - S'$ only has odd components which are all factor-critical and with S' matchable to $\mathcal{C}_{G-S'}$.
Note that, as every maximum matching does not cover exactly $q(G - S') - |S'|$ vertices by the stronger version of Tutte's Theorem, Theorem 2.2.3 implies that every maximum matching M covers all vertices in S' and never contains an edge between vertices in S'. Concretely, every $s \in S'$ is matched to a vertex of a component in $\mathcal{C}_{G-S'}$ where no two distinct vertices in S' are matched to vertices of the same component. Those components that contain a vertex matched to a vertex in S' are completely covered by M and those that do not have exactly one vertex that is not covered by M.
Now, our goal is to iteratively shrink S' to a desired S: Assume that $X \subseteq S'$ is not strongly matchable, i.e. $\left|N_{G_{S'}}(X)\right| = |X|$, where $G_{S'}$ denotes the bipartite graphs with partition classes S' and the set of odd components of $G - S'$ as in the preliminary of Theorem 2.2.3. Then X together with the odd components in $N_{G'_S}(X)$ induce an even subgraph[4] in G which, as every component in $N_{G'_S}(X)$ is factor-critical and X is matchable to $N_{G'_S}(X)$, has a perfect matching. Furthermore, $S' \setminus X$ is matchable to $\mathcal{C}_{G-(S' \setminus X)}$. Indeed, if that was not the case, then there exists $Y \subseteq S' \setminus X$ such that $\left|N_{G_{S' \setminus X}}(Y)\right| < |Y|$. But this contradicts S' being matchable to $\mathcal{C}_{G-S'}$ as

$$\left|N_{G_{S'}}(X \cupdot Y)\right| = \left|N_{G_{S'}}(X)\right| + \left|N_{G_{S' \setminus X}}(Y)\right| < |X| + |Y| = |X \cupdot Y| . \lightning$$

Furthermore, as the components in $\mathcal{C}_{G-(S' \setminus X)}$ are exactly those components in $\mathcal{C}_{G-S'}$ that were in $G_{S'}$ not adjacent to X, we get that $S' \setminus X$ also has maximum deficiency and that every component in $\mathcal{C}_{G-(S' \setminus X)}$ is factor-critical. Moreover, note that by our observation on how every maximum matching M is structured, we have that every maximum matching must cover all vertices contained in the even components of $G - (S' \setminus X)$, every $s \in S' \setminus X$ is matched to a vertex of a component in $\mathcal{C}_{G-S'}$ where no two distinct vertices in S' are matched to vertices of the same component. Those odd components that contain a vertex matched to a vertex in S' are completely covered by M and those that do not have exactly one vertex that is not covered by M.
We may apply the above reduction iteratively until the resulting set $S \subseteq S'$ is strongly matchable to $\mathcal{C}_{G-S}$ and has all the other desired results.
For the uniqueness, we consider the vertices contained in even components of $G - S$, those contained in odd components of $G - S$ and those in S separately, where we will refer to them as being of *type 1*, *type 2* and *type 3* in that order. As discussed, type 1 and 3 vertices are always covered by a maximum matching. Consider now a vertex $v \in V(G)$ of type 2. In $G - v$, the component containing v becomes an even component in $G - v - S$ and has by the factor-criticaliy a perfect matching. Furthermore, as S is strongly matchable to $\mathcal{C}_{G-S}$, S is also matchable to $\mathcal{C}_{G-v-S}$, so we can find a matching covering S with one endpoint in S and the other in $\mathcal{C}_{G-v-S}$ where – again – no two vertices in S get matched to vertices in the same odd component. Thus, by the factor-criticality of all of the odd components that did not contain v, which are exactly the

[4]As in every connected component in the subgraph is of even order.

components in $\mathcal{C}_{G-v-S}$, and the fact that all even components have a perfect matching, we can extend the matching covering S to a maximum matching M of $G - v$. As it is of the same size as a maximum matching of G by construction, we get that M is a maximum matching of G that doesn't contain v. Hence, for every type 2 vertex v there exists a maximum matching that does not contain v.
Thus, we conclude that type 1 vertices are exactly the vertices that are contained in every maximum matching and are not adjacent to any vertex of type 2, type 2 vertices are exactly the vertices for which there exists a maximum matching that does not contain them, and type 3 – or vertices in S – are exactly those vertices that are contained in every maximum matching but are adjacent to a vertex of type 2.
This completes the proof. $\square$

Exercise 3

Let T be a tree and let $\mathcal{T}$ be a set of subtrees of T. Then the maximum number of disjoint[5] trees in $\mathcal{T}$ equals the least cardinality of a set X of vertices such that $T - X$ contains no tree from $\mathcal{T}$.

Proof. We will do an induction on the number of vertices $n := |V(T)|$. If $n = 1$, then the claim is clear. So, consider $n > 1$, meaning that T has at least one leaf v.
We will say that $X \subseteq V(T)$ *hits* a tree $H \subseteq T$ if $X \cap V(H) \neq \emptyset$. Note that X hits all trees in $\mathcal{T}$ if and only if $T - X$ contains no tree from $\mathcal{T}$. We do a case distinction:

Case 1: There is a subtree $H \in \mathcal{T}$ that only consists of a leaf v in T, i.e. $H = (\{v\}, \emptyset)$. Then, to hit H, any X as above must contain v. So, it is sufficient and necessary for $X \setminus \{v\}$ to hit all trees not containing v. Hence, consider

$$T' := T - v,$$
$$\mathcal{T}' := \{H \in \mathcal{T} \mid v \notin V(H)\}.$$

Let $\mathcal{M}' \subseteq \mathcal{T}'$ be a maximum set of disjoint trees and $X' \subseteq V(T')$ be a least cardinality set such that $T' - X'$ contains no tree from $\mathcal{T}'$. By the induction hypothesis, $|\mathcal{M}'| = |X'|$. Now, let $\mathcal{M} := \mathcal{M}' \mathbin{\dot\cup} \{H\}$ and $X = X' \mathbin{\dot\cup} \{v\}$.

As X' hits all trees in $\mathcal{T}'$ and is minimum in size, it is clear that X as defined is a least cardinality set hitting all trees in $\mathcal{T}$. So, it remains to show that $\mathcal{M}$ is a maximum set of disjoint trees of $\mathcal{T}$: Note that any maximum set $\mathcal{N}$ of disjoint trees of $\mathcal{T}$ contains a tree containing v as otherwise we could add to the set H. Furthermore, we may assume that H is in that maximum set of disjoint trees, as otherwise we could replace the tree in Y containing v by H. So, for $\mathcal{N} \setminus \{H\}$ to be optimal, it is necessary and sufficient to consider exactly the trees in $\mathcal{T}'$. Therefore, $\mathcal{M}$ as constructed is indeed optimal.

[5] As in vertex-disjoint.

Case 2: $(\{v\}, \emptyset) \notin \mathcal{T}$. Then let

$$T' := T - v,$$
$$\mathcal{T}' := \{H - v \mid H \in \mathcal{T}\}.$$

As v is a leaf, T' and the elements of $\mathcal{T}'$ remain trees. Furthermore, let $\mathcal{M}' \subseteq \mathcal{T}'$ be a maximum set of disjoint trees and $X \subseteq V(T')$ be a least cardinality set such that $T' - X$ contains no tree from $\mathcal{T}'$. By the induction hypothesis, $|\mathcal{M}'| = |X|$. Let $\mathcal{M}$ be the set of trees in T corresponding to the trees in $\mathcal{M}'$.[6] Note that $\mathcal{M}$ has the same cardinality as $\mathcal{M}'$ as the trees in $\mathcal{M}'$ are disjoint.

We now claim that $\mathcal{M}$ and X also satisfy their respective properties for T.

Indeed, as v is a leaf, any two trees in $\mathcal{T}$ that contain v must also contain v's unique neighbor $u \in V(T)$. As u is contained in at most one tree of $\mathcal{M}'$, $\mathcal{M}$ can contain at most one tree containing v and is thus also a set of disjoint trees. It is also clear that $\mathcal{M}$ is optimal in size as applying the mapping $H \mapsto H \setminus \{v\}$ will always lead to a maximum set of disjoint trees of T' of the same cardinality.

It is also clear that X hits every tree in $\mathcal{T}$, as it hits every tree in $\mathcal{T}'$ by assumption. Furthermore, X is also optimal in size with respect to $\mathcal{T}$ as any Y hitting all trees in $\mathcal{T}$ also hits all trees in $\mathcal{T}'$.

This completes the proof. □

Remark. The following infinite analogue can be shown via transfinite induction: Let T be on infinite tree that does not contain infinite paths and let $\mathcal{T}$ be a family of subtrees. Then there is a disjoint $\mathcal{T}' \subseteq \mathcal{T}$ such that one can choose exactly one vertex from each $H \in \mathcal{T}'$ such that the resulting set X meets every element of $\mathcal{T}$.

Exercise 4

Every graph, which does not contain two vertex-disjoint cycles, can be turned into a forest by removing at most three vertices.

Proof. Let G be a (with respect to the size of $V(G)$) minimal counterexample to the claim, i.e. G is a graph with no two vertex-disjoint cycles such that one needs to remove four vertices for G to turn into a forest. Clearly, G is non-empty and contains cycles.

1. G is connected. Otherwise, if $C^1, \ldots, C^m, m > 1$, are the connected components of G, exactly one of the C^i contains cycles, so we get a smaller counterexample. ↯
2. G is C_3-free: If G contains a C_3, deleting the vertices on that cycle would, by assumption, create a forest. ↯
3. $\delta(G) \geq 3$: Assume otherwise. Clearly, G can't contain isolated vertices or leaves due to minimality. So, let v be of degree 2 with neighbors u, w. Note that $uw \notin E(G)$

[6] If $H \in \mathcal{T}$ with $v \in H$ and $H - v \in \mathcal{T}$, then choose $H - v$ for $\mathcal{M}$.

since G is C_3-free. Consider the graph $H = G - v + \{uw\}$ which still has no two vertex-disjoint cycles. Furthermore, as $G - \{x, y, z\}$ is not a forest for all $x, y, z \in V(H)$, the same is true for $H - \{x, y, z\}$. So G is not minimal. ↯

4. Moreover, G needs to be C_4-free: Let G contain a C_4 induced by vertices v_1, v_2, v_3, v_4 (in that order). Denote this cycle by C. Since G doesn't contain two vertex-disjoint cycles, every cycle in G must intersect with C and $G - C$ must be a forest that is non-empty due to 2.
We first observe that no vertex $v \in V(G-C)$ can be adjacent to vertices v_i, v_{i+1} for some $i \in [4]$ as this would induce a triangle. In particular, no vertex $v \in V(G - C)$ can be adjacent to three vertices of C and if v is adjacent to two vertices of C, they need to be v_i, v_{i+2} for some $i \in [2]$.
As $G-C$ is a non-empty forest, it must contain a vertex v with $\deg_{G-C}(v) \leq 1$. To satisfy $\delta(G) \geq 3$, we actually must have $\deg_G(v) = 3$ and that v must be adjacent to exactly two non-adjacent vertices of C, so w.lo.g. $N_G(v) \cap V(C) = \{v_1, v_3\}$.
As G is a counterexample, $G - \{v_1, v_2, v_3\}$ must contain a cycle C'. As C' needs to intersect C, $V(C') \cap V(C) = \{v_4\}$.
Note however that $v \notin V(C')$. Indeed, obviously we have $v \neq v_4$ and by definition that v is not adjacent to v_4. So, if $v \in V(C')$, v would have to be adjacent to two distinct vertices in $V(G - C)$, which is also not the case due to $\deg_{G-C}(v) \leq 1$.
However, this means that C' and $v_1v_2v_3vv_1$ are two vertex-disjoint cycles in G. ↯

5. G doesn't exist: Let C be a shortest cycle in G. Note that by the above, $|C| > 4$.
Now, every vertex $V - C$ has at most one neighbor in C. Indeed, if $v \in V - C$ has two neighbors u, w, then either one of the u-w-segments of C is of length 1, so with uvw would create a triangle, or one of its segments has at most length $|C|/2$, so the segment together with uvw creates a cycle of length $|C|/2 + 2 < |C|$. ↯
Now, if we were to consider $G-C$, note that $\delta(G-C) \geq 2$, so $G-C$ must contain a cycle, contradicting G containing no two vertex-disjoint cycles. ↯

This concludes our proof. □

Sheet 2

Exercise 1

The edge set of every outerplanar graph is the union of two forests.

Proof. We first recall some facts about planar graphs:

- Euler's formula asserts that if G is a connected, planar graph with n vertices, m edges and f faces, then $n - m + f = 2$.
- By counting the number of edge-face-incidences, we get $3f \leq 2m$, so plugging that into Euler's formula, we have for G as above

$$3 \cdot (n - m - 2) \leq 2m \implies m \leq 3n - 6.$$

- Let G now be an outerplanar graph. Consider the graph H where we add to G a vertex which is adjacent to every other vertex of G. Note that H is planar and connected with $|E(G)| + |V(G)|$ edges and $|V(G)| + 1$ vertices. So,

$$|E(G)| + |V(G)| \leq 3 \cdot (|V(G)| + 1) - 6 \implies |E(G)| \leq 2 \cdot |V(G)| - 3.$$

So, again, let G be an outerplanar graph. Note that the drawing of G in which every vertex lies on the boundary also naturally induces a planar drawing of $G[U]$ in which every vertex of U lies on the boundary for every $U \subseteq V$. This shows that being outerplanar is hereditary with respect to the subgraph-relation. By our bound above for the number of edges for planar graphs, we get

$$|E(G[U])| \leq 2 \cdot |U| - 3 \leq 2 \cdot (|U| - 1)$$

for every $U \subseteq V$. Thus, by Theorem 2.4.3 (Nash-Williams 1964), we get that G can be covered by at most 2 trees F_1 and F_2. By suitably deleting from F_1 and F_2 edges that are not in G, we thus get that G is the (edge-disjoint) union of forests F_1 and F_2. $\square$

Exercise 2

If G is a k-linked graph, we have that

(*i*) G is $(2k-1)$-connected, and

(*ii*) if $s_1, \ldots, s_k, t_1, \ldots, t_k$ are not necessarily distinct vertices of G such that $s_i \neq t_i$ for all $i \in [k]$, then G contains independent s_i-t_i-paths P_i for $i \in [k]$.

Proof. Let G be a k-linked graph.

(*i*) Assume not, i.e. there are $2k-2$ distinct vertices $s_1, \ldots, s_{k-1}, t_1, \ldots, t_{k-1}$ such that their deletion disconnects G. Denote the set of those $2k-2$ vertices by S and consider $s_k, t_k \in V(G-S)$ of different connected components of $G-S$. As G is k-linked, there are vertex-disjoint s_i-t_i-paths P_i for every $i \in [k]$. However, the path P_k must exist in $G-S$ due to the vertex-disjointness of the paths. ↯

(*ii*) Let $S = (s_1, \ldots, s_k, t_1, \ldots, t_k)$. Denote by N the number of vertices that appear more than once in S. We will do an induction on N.
For the induction base, consider $N = 0$. In other words, the vertices in S are distinct. Hence, the existence of those paths follows directly from the k-linkedness.
For the induction step, assume $N > 0$. Let $v \in V(G)$ be such a vertex that appears $r > 1$ times. Since we have $s_i \neq t_i$ for all $i \in [k]$, r is at most k. By (i), we know that G is $(2k-1)$-connected, so in particular $|N(v)| \geq 2k-1$.
We will now count how many other vertices can appear at most in S: From each pair that does't involve v, we gain at most two vertices, and for each involving v we get at most one. Hence, there are at most $2(k-r) + r = 2k + r$ other vertices. So, there are at least

$$(2k-1) - (2k+r) = r - 1$$

neighbors of v that do not appear in S.
Hence, consider the sequence $s'_1, \dots, s'_k, t'_1, \dots, t'_k$ where we replaced $r-1$ occurrences of v by those neighbors such that v and those neighbors appear only once in the sequence. This new sequence thus has $N-1$ vertices that appear multiple times, i.e., by applying the induction hypothesis, we get s'_i-t'_i-paths P'_i that are independent. In particular, since v appears in one of the paths as endpoint where v is distinct in the sequence, none of the other paths contain v. So, the paths, where one endpoint is a previously chosen neighbor of v, can be extended to v. These new resulting paths $(P_i)_{i\in[k]}$ are now independent s_i-t_i-paths.

This completes the proof. $\square$

Exercise 3

Every k^2-linked graph contains a TK_k.

Proof. If $k \in \{1, 2\}$, the statement is clear. For $k > 2$, we shall prove a stronger statement, specifically that $\binom{k}{2}$-linkedness is sufficient: Let G be $\binom{k}{2}$-linked. By definition, G contains $2\cdot\binom{k}{2} = k\cdot(k-1) > k$ vertices. So, take arbitrary vertices $v_1, \dots, v_k$. Applying Exercise 2 (ii), where for every edge in $\{i, j\} \in \binom{[k]}{2}$ (let's say that this is the l-th edge enumerating $\binom{[k]}{2}$) we add $s_l = v_i, t_l = v_j$, we get that there are independent v_i-v_j-paths for all $\{i, j\} \in \binom{[k]}{2}, i < j$. The graph formed by the union of these paths is a TK_k. $\square$

Exercise 4

(i) For every $k \in \mathbb{N}$ there exists a $(3k-3)$-connected graph that isn't k-linked.

Proof. Let G be a graph with vertices $s_1, \dots, s_k, t_1, \dots, t_k$ and additional $k-1$ vertices, so $|G| = 3k-1 > 3k-3$. Furthermore, let

$$E(G) = \binom{V(G)}{2} - \{s_i t_i \mid i \in [k]\}.$$

By Menger's Theorem, to prove $(3k-3)$-connectedness, it suffices to show that $\forall\, a, b \in V(G), a \neq b$, we have $3k-3$ independent a-b-paths.
By construction, every such pair a, b is of the form s_i, t_i for some $i \in [k]$. But again by construction, we know that all other $(3k-1)-2 = 3k-3$ are adjacent to both of these vertices, so in particular we find $3k-3$ independent a-b-paths. However, G is not k-linked, as there are no vertex-disjoint s_i-t_i-paths for all $i \in [k]$ since every such path needs to use one of the $k-1$ additional vertices. This completes the proof. $\square$

(ii) There is a sequence of graphs $(G_n)_{n\in\mathbb{N}}$ such that $|V(G_n)| \to \infty$ as $n \to \infty$ and G_n is 5-connected but not 2-linked for every $n \in \mathbb{N}$.

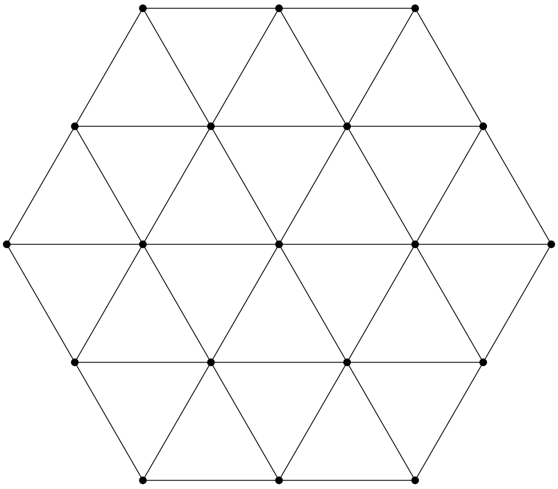

Figure 1: H_1

Proof sketch (given by Henri in class). Consider a sequence of graphs H_n with H_1 given by Figure 1 and H_{i+1} by subdividing each edge in H_i and triangulating the interior of the "hexagon" such that each individual triangle becomes a "triforce". For G_n, take two copies of H_n and add six vertices such that each vertex is adjacent to every vertex on one "side" of each copy as in Figure 2. To show that this graph is not 2-linked, take as s_1 and t_1 the left outmost / right outmost vertex and for s_2 and t_2 the uppermost / lowermost vertex as in Figure 2. "Topologically", one can see that it is not possible to have any two disjoint s_1-t_1- and s_2-t_2-paths.

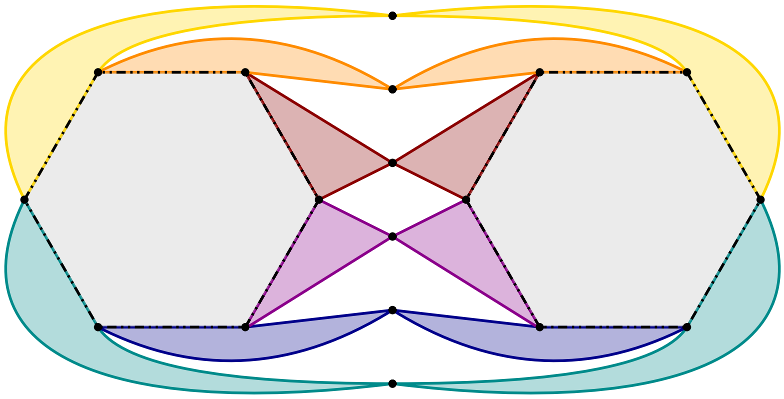

Figure 2: Sketch for G_n

The proof that G_n is 5-connected is left as an exercise for the reader. □

Remark. Really, what this proof boils down to is that *any* 5-connected planar graph where one face is not bounded by a triangle is not 2-linked.

Sheet 3

Exercise 1 (Gallai's Theorem)

The edge set of any graph G can be written as a disjoint union $E(G) = B \mathbin{\dot\cup} C$ with C and B being elements from the cycle space $\mathcal{C}(G)$ and the cut space $\mathcal{B}(G)$, respectively.

Proof. As elements of $\mathcal{C}$ induce exactly the subgraphs with vertex V where every vertex has even degree[7], it suffices to show that we can partition V into V_1, V_2 such that $G[V_i]$ is an even graph for $i \in [2]$. Then we can take $C = E(G[V_1]) \mathbin{\dot\cup} E(G[V_2])$ and $B = E(V_1, V_2)$. Note that, unlike the usual convention, we allow V_1 or V_2 to be empty.
We will now proceed by induction on the number of vertices n. To indicate with respect to which graph we consider certain quantities, we will write the graph in the subscript where necessary. Clearly, the claim is true for $n = 1$, so consider $n > 1$. If all vertices in G have even degree, we can take $V_1 = V$ and $V_2 = \emptyset$. Otherwise, there exists $v \in V(G)$ with odd degree. Define G' with $V(G') = V(G) \setminus \{v\}$ and

$$E(G') = E_G(N_G(v), V(G') - N_G(v)) \mathbin{\dot\cup} \left(\binom{N_G(v)}{2} \setminus E(G[N_G(v)]) \right).$$

In other words, G' has the same adjacencies as G except between neighbors of v, where we invert their adjacency. Now, by induction hypothesis, there is $V_1' \mathbin{\dot\cup} V_2' = V(G')$ such that $G'[V_i']$ is even for all $i \in [2]$. As $|N_G(v)|$ is odd, exactly one of $|V_1' \cap N_G(v)|$ and $|V_2' \cap N_G(v)|$ is odd, w.l.o.g. $V_2' \cap N_G(v)$.
We now claim that $V_1 = V_1' \mathbin{\dot\cup} \{v\}$ and $V_2 = V_2'$ is a desired partition:
Clearly, v has even degree in $G[V_1]$. As we didn't change the neighborhoods for vertices not in $N_G(v) \cup \{v\}$, it suffices to show that every vertex $w \in N_G(v)$ has in its assigned partition even degree:

1. $w \in N_G(v) \cap V_1$: Let $N_1 = N_G(v) \cap V_1 = N_G(v) \cap V_1'$. As w's neighborhood got inverted, we find

   $$E_{G'[V_1']}(\{w\}, N_1) \mathbin{\dot\cup} E_{G[V_1]}(\{w\}, N_1) = N_1 \setminus \{w\}.$$

   So, since $\deg_{G'[V_1']}(w)$ is even and $|N_1|$ is even, we get

   $$\begin{aligned} \deg_{G[V_1]}(w) &= 1 + \left|E_{G[V_1]}(\{w\}, N_1)\right| + \left|E_{G[V_1]}(\{w\}, V_1' \setminus N_1)\right| \\ &= 1 + (|N_1| - 1) - \left|E_{G'[V_1']}(\{w\}, N_1)\right| + \left|E_{G'[V_1']}(\{w\}, V_1' \setminus N_1)\right| \\ &\equiv |N_1| + \left|E_{G'[V_1']}(\{w\}, N_1)\right| + \left|E_{G'[V_1']}(\{w\}, V_1' \setminus N_1)\right| \\ &\equiv \deg_{G'[V_1']}(w) \\ &\equiv 0 \pmod 2. \end{aligned}$$

[7] Proposition 1.9.1 in Diestel's *Graph Theory*.

2. $w \in N_G(v) \cap V_2$: Let $N_2 = N_G(v) \cap V_2 = N_G(v) \cap V_2'$. As w's neighborhood got inverted, we find

$$E_{G'[V_2']}(\{w\}, N_2) \cup E_{G[V_2]}(\{w\}, N_2) = N_2 \setminus \{w\}.$$

So, since $\deg_{G'[V_2']}(w)$ is even and $|N_2|$ is odd, we get

$$\begin{aligned}
\deg_{G[V_2]}(w) &= \left|E_{G[V_2]}(\{w\}, N_2)\right| + \left|E_{G[V_2]}(\{w\}, V_2 \setminus N_2)\right| \\
&= (|N_2| - 1) - \left|E_{G'[V_2']}(\{w\}, N_2)\right| + \left|E_{G'[V_2']}(\{w\}, V_2' \setminus N_2)\right| \\
&\equiv \left|E_{G'[V_2']}(\{w\}, N_2)\right| + \left|E_{G'[V_2']}(\{w\}, V_2' \setminus N_2)\right| \\
&\equiv \deg_{G'[V_2']}(w) \\
&\equiv 0 \pmod 2.
\end{aligned}$$

This proves the claim. □

Remark. Interestingly enough, this theorem is not (to my knowledge) an immediate consequence of the cycle space and bond space being orthogonal to each other as – at least when we look at vector spaces over a finite field – orthogonal spaces may non-trivially intersect.

Exercise 2

(i) In a connected graph the minimal edge sets containing an edge from every spanning tree are precisely its bonds.

Proof. Let G be a connected graph. We know that in this situation the bonds are precisely the minimal cuts of G. Call an edge set *good* if it contains an edge from every spanning tree.
We first show that an edge set $C \subseteq E(G)$ is a minimal cut if and only if $G - C$ has exactly two connected components: For the one direction, let C be a minimal cut. Then, there is a partition (with non-empty partition classes) V_1, V_2 such that $C = E(V_1, V_2)$. Suppose that $G[V_1]$ is disconnected. Let $V_1' \subseteq V_1$ be a subset such that $G[V_1']$ is a connected component in $G[V_1]$. As G is connected, $E(V_1', V_2) \neq \emptyset$. So, $E(V_1 - V_1', V_1' + V_2)$ is a proper subset of C. ↯
For the other direction, suppose that C is an edge set such that $G - C$ contains exactly two components. Let V_1, V_2 be the vertex sets of those components. By construction, we have $C = E(V_1, V_2)$, so C is a cut. It is also minimal as any proper subset of C would connect the two components in $G - C$ such that the resulting graph stays connected and hence could not be a cut.
It remains to show that $G - C$ contains exactly two components if and only if C is a minimal good edge set: For the one direction, suppose that C contains exactly two components. As every spanning tree of G needs to connect the vertices of V_1 and V_2, they all must contain an edge of C. It is also clear that C would be minimal by a similar argument as above.

For the other direction, suppose that C is a minimal good edge set. Clearly, removing C disconnects G (otherwise $G - C$ would contain a spanning tree of G). Let the connected components of $G - C$ be $G_1, \ldots, G_k, k \geq 2$. As G is connected, there exists $j \in [k-1]$ such that $E(G_j, G_k) \neq \emptyset$. If $k > 2$, then consider $C' = C - E(G_j, G_k)$. $G - C'$ would have exactly $k - 1 \geq 2$ components, so C' is a proper good subset of C, contradicting C being minimal. ↯
This proves the claim. □

(ii) There are graphs for which the cycle space is not generated by its cycles and cuts.

Proof. Consider K_4. Label the vertices $v_1, \ldots, v_4$. Clearly, $E(\{v_1, v_3\}, \{v_2, v_4\})$ is the 4-cycle $C = v_1 v_2 v_3 v_4 v_1$. This means that $C \in \mathcal{B}(K_4) \cap \mathcal{C}(K_4)$, so $\dim(\mathcal{B}(K_4) \cap \mathcal{C}(K_4)) \geq 1$. In particular, we have

$$\begin{aligned}
\dim(\mathcal{B}(K_4) + \mathcal{C}(K_4)) &= \dim(\mathcal{B}(K_4)) + \dim(\mathcal{C}(K_4)) - \dim(\mathcal{B}(K_4) \cap \mathcal{C}(K_4)) \\
&= \dim(\mathcal{B}(K_4)) + \dim(\mathcal{B}(K_4)^{\perp}) - \dim(\mathcal{B}(K_4) \cap \mathcal{C}(K_4)) \\
&\leq \dim(\mathcal{E}(K_4)) - 1.
\end{aligned}$$

Note that the second equality follows from the bond space and cycle space being orthogonal to each other. So, $\mathcal{E}(K_4)$ is not generated by its cycles and cuts. □

Exercise 3

A 2-connected plane graph G is bipartite if and only if every face is bounded by an even cycle.

Classical proof. Let G be a 2-connected plane graph. We know that

- a graph is bipartite if and only if every cycle is of even length,
- a plane graph is 2-connected if and only if every face is bounded by a cycle.

Hence, if G is bipartite, it directly follows that every face is bounded by an even cycle. For the other direction, assume that there is an odd cycle in G despite every face being bounded by an even cycle. Let C be an odd cycle with the smallest number of interior faces. The interior must contain at least 2 faces (otherwise C bounds a face ↯). By 2-connectedness, there exists P with endpoints on C and edges in the interior of C. Let C_1, C_2 be the cycles in $C \cup P$ different from C. It follows that

$$|E(C_1)| + |E(C_2)| = \underbrace{2\,|E(P)|}_{\text{even}} + \underbrace{|E(C)|}_{\text{odd}}.$$

This implies that exactly one of C_1, C_2 is an odd cycle, w.l.o.g. C_1. But C_1 has fewer faces in its interior than C. ↯ □

Linear Algebra proof. Again, if G is bipartite, it directly follows that every face is bounded by an even cycle. So, let G be a 2-connected plane graph where every face is bounded by an even cycle. Let C be an arbitrary cycle in G. We need to show that C is of even length. For that, let $f_1, \ldots, f_l$ be the interior faces of C and $e_i \in \mathcal{C}(G)$ denote the edge set of the boundary cycle C_i of f_i. Similary, let $e_C = E(C) \in \mathcal{C}(G)$. Note that

$$e_C = \sum_{i=1}^{l} e_i.$$

So, as every f_i is bounded by an even cycle, we have

$$|E(C)| \equiv \langle e_C, E(G)\rangle \equiv \sum_{i=1}^{l} \langle e_i, E(G)\rangle \equiv \sum_{i=1}^{l} |C_i| \equiv 0 \pmod 2.$$

So C is even and G bipartite. □

Exercise 4 (Euler's Formula)

Every 2-connected plane graph G satisfies Euler's Formula, i.e.

$$|V(G)| - |E(G)| + |F(G)| = 2.$$

Proof. Let G be a 2-connected plane graph with vertex set V, edge set E and face set F. Let $\mathcal{F}$ be the face space (over $\mathbb{F}_2$), defined over the power set of F with addition given by the symmetric difference. Furthermore, let the boundary map $\varphi\colon \mathcal{F} \to \mathcal{E}$ be the linear map given by

$$\varphi(\{f\}) = \{e \in E \mid e \text{ is incident to } f.\}.$$

Since $\{\{f\} \mid f \in F\}$ forms a basis of $\mathcal{F}$, φ is well-defined.
We will first calculate the kernel of φ: Let $M \subseteq F$ with $\varphi(M) = 0$. By

$$\varphi(M) = \sum_{f \in M} \varphi(\{f\}),$$

we know that $\varphi(M) = 0$ is exactly the case if every edge $e \in E$ appears in an even number of faces in M. Since G is plane, every edge appears must therefore appear in two or zero faces in M. Hence, $M \in \{\emptyset, F\}$. Indeed, if M would be a non-empty, proper subset of F, then consider a maximal set $N \subseteq F \setminus M$ such that in the dual graph of G they would be connected. Since $M \neq \emptyset$ is a proper subset of F, N is a non-empty proper subset of F as well. Hence, there must be edges that bound both a face in M and a face in N. However, those edges appear in exactly one face in M. ↯
Thus, $\ker(\varphi) = \langle \{F\} \rangle$.
Next, we analyze $\varphi(\mathcal{F})$: Let $f \in F$. As each face is bounded by a cycle, we have that $\varphi(\{f\}) \in \mathcal{C}$. As $\{\{f\} \mid f \in F\}$ forms a basis of $\mathcal{F}$, we have that $\varphi(\mathcal{F}) \subseteq \mathcal{C}$. To get equality, we show that every cycle is in the image of φ.

For that we will proceed by induction over the number of interior faces N of cycle C: If $N = 1$, then C is a cycle bounding a face f, hence $\varphi(\{f\}) = C$. Now, let $N > 1$ and let $f \in F$ be an interior face of C such that f is incident to edges in C. Let C_f be the unique cycle bounding f. Then, $C' = C + C_f$ defines a cycle with $N - 1$ interior faces, so by the induction hypothesis, it is in the image of φ. As both C' and C_f are in the image of φ, we have that C is also in the image of φ.
Now, by the dimension formula for linear maps and Theorem 1.9.5, we get

$$|F| = \dim(\ker(\varphi)) + \operatorname{rank}(\varphi) = 1 + \dim(\mathcal{C}) = |E| - |V| + 2 \implies |V| - |E| + |F| = 2.$$

This completes the proof. $\square$

Remark. Using this, one can show that all connected plane graphs G satisfy Euler's Formula: If G contains cutvertices $v_1, \ldots, v_j$, consider the blocks $B_1, \ldots, B_l$ of G. W.l.o.g. let all blocks of G be incident to the outer face. Note that the block-cutvertex graph of G being a tree implies the existence of such an embedding.
Each block would either be a 2-connected plane graph or a bridge, where the latter obviously satisfies Euler's Formula. Now, clearly $E(B_1) \cupdot \ldots \cupdot E(B_l) = E(G)$ and the blocks pairwise "share" exactly one face, namely the outer face.
Furthermore, as the vertices that appear in multiple blocks are exactly the cutvertices, each cutvertex v_i would contribute in the sum $\sum_{k=1}^{l} |V(B_k)|$ exactly $\deg_{\mathcal{G}}(v_i)$ times, so

$$\begin{aligned} |V| &= \sum_{k=1}^{l} |V(B_k)| - \sum_{i=1}^{j} (\deg_{\mathcal{G}}(v_i) - 1) \\ &= \sum_{k=1}^{l} |V(B_k)| - (|E(\mathcal{G})| - j) \\ &= \sum_{k=1}^{l} |V(B_k)| - ((j + l - 1) - j) \\ &= \sum_{k=1}^{l} |V(B_k)| - (l - 1), \end{aligned}$$

where $\mathcal{G}$ is the block-cutvertex graph of G. The second equality follows from all edges in $\mathcal{G}$ being between cutvertices and blocks and the third equality follows from $\mathcal{G}$ being a tree. So, putting all of this together, we get

$$\begin{aligned} |V(G)| - |E(G)| + |F(G)| &= \sum_{k=1}^{l} |V(B_k)| - (l - 1) - \sum_{k=1}^{l} |E(B_k)| + \sum_{k=1}^{l} |F(B_k)| - (l - 1) \\ &= \sum_{k=1}^{l} (|V(B_k)| - |E(B_k)| + |F(B_k)|) - 2(l - 1) \\ &= 2. \end{aligned}$$

Sheet 4

Exercise 1

For a graph $G = (V, E)$ let $A(G)$ denote the adjacency matrix of G and let $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_{|V|}(G)$ denote the eigenvalues of $A(G)$.
Then $\sum_{i=1}^{|V|} \lambda_i^k(G)$ is the number of closed walks of length k for every $k \geq 0$.

Proof. For the sake of brevity, we will set $A = A(G)$ and sometimes call a walk of length k a k-walk.

(1) $(A^k)_{u,v}$ is the number of u-v-walks of length k for $u, v \in V$. We will show this by induction on $k \geq 0$: If $k = 0$ or $k = 1$, the claim is clear. So, consider $k > 1$.
For vertices $u, w \in V$, we have that

$$(A^{k+1})_{u,w} = \sum_{v \in V} (A^k)_{u,v} \cdot A_{v,w}.$$

By the induction hypothesis, $(A^k)_{u,v}$ is exactly the number of u-v-walks of length k. So, as such a walk is extendable to a u-w-walk of length $k+1$ if and only if $vw \in E$, we see that $(A^{k+1})_{u,w}$ counts the number of u-w-walks of length $k+1$ as claimed.

(2) By (1), $\operatorname{tr}(A^k)$ therefore counts the number of closed k-walks.

(3) $\operatorname{tr}(A^k) = \sum_{i=1}^{|V|} \lambda_i^k(G)$ for all $k \geq 0$: Since A is real and symmetric, there exists by the Spectral Theorem an orthogonal matrix $U \in \mathbb{R}^{V \times V}$ such that

$$A = U \cdot \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{|V|}) \cdot U^\top.$$

In particular, this also means that

$$\begin{aligned} A^k &= \left(U \cdot \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{|V|}) \cdot U^\top\right)^k \\ &= U \cdot \left(\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{|V|})\right)^k \cdot U^\top \\ &= U \cdot \operatorname{diag}(\lambda_1^k, \lambda_2^k, \ldots, \lambda_{|V|}^k) \cdot U^\top. \end{aligned}$$

As the trace of a matrix is a similarity-invariant, we in particular get

$$\operatorname{tr}(A^k) = \sum_{i=1}^{|V|} \lambda_i^k(G).$$

This completes the proof. $\square$

Remark. There was a discussion whether you could also interpret the quantity as k times the number of closed k-walks that are (ignoring the starting point) equal. But this doesn't work, as for example the closed k-walk by just repeatedly walking back and forth on an edge doesn't give you k different closed walks.

Exercise 2

We have $\mathrm{ch}(K_{2,\dots,2}) = n$ for the complete n-partite graph with two vertices in each class.

Proof. We prove the statement by induction on n. Denote the complete n-partite graph with two vertices in each class by G_n. We may relabel the vertices of G_n such that

$$V(G_n) := \{a_i : i \in [n]\} \mathbin{\dot\cup} \{b_i : i \in [n]\} \text{ and } E(G_n) := \binom{V(G_n)}{2} \setminus \{a_i b_i : i \in [n]\}.$$

Consider $n = 1$ for the induction base. As $E(G_1) = \emptyset$, we trivially have $\mathrm{ch}(G_1) = 1$. Assume now that the claim holds for a fixed, but abitrary $n \in \mathbb{N}$ and consider G_{n+1}. Clearly, as $G_{n+1}[\{a_i : i \in [n+1]\}] \simeq K_{n+1}$, each a_i needs to be assigned a different color for a proper coloring. Hence, we have that G_{n+1} is not L-list-colorable with $L(v) = [n]$ for all $v \in V(G_{n+1})$. Therefore, $\mathrm{ch}(G_{n+1}) \geq n+1$. To show equality, it suffices to show that G_{n+1} is $(n+1)$-list-colorable:
Let L be an arbitrary list of colors such that $|L(v)| = n+1$ for all $v \in V(G_{n+1})$.

Case 1: $\exists\, i \in [n+1] \colon L(a_i) \cap L(b_i) \neq \emptyset$. Let $x \in L(a_i) \cap L(b_i)$. Define the list of colors $L' \colon V', v \mapsto L(v) \setminus \{x\}$ with $V' := V(G_{n+1}) \setminus \{a_i, b_i\}$. Note that $|L'(v)| \geq n$ for all $v \in V'$. Furthermore, we have that

$$E(G_{n+1}[V']) = \binom{V'}{2} \setminus \{a_j b_j : j \in [n+1] \setminus \{i\}\} \implies G_{n+1}[V'] \simeq G_n.$$

So, applying the induction hypothesis, there exists a coloring c' with $c'(v) \in L'(v) \subseteq L(v)$ for all $v \in V'$. Using c', define

$$c \colon V(G_{n+1}) \to \mathbb{N}, v \mapsto \begin{cases} x, & v \in \{a_i, b_i\} \\ c'(v), & \text{otherwise.} \end{cases}$$

Indeed, c is a proper L-coloring since none of the $v \in V'$ are colored with x, a_i and b_i are not adjacent and $c(v) \in L(v)$ for all $v \in V(G_{n+1})$.

Case 2: $\forall\, i \in [n+1] \colon L(a_i) \cap L(b_i) = \emptyset$. If Case 2 is the case, define a bipartite graph $F = (A \cup B, E(F))$ with $A = V(G_{n+1}), B = \bigcup_{v \in A} L(v)$ and let $uc \in E(F)$ if and only if $c \in L(u)$. Observe that G_{n+1} has an L-list-coloring if and only if there is a matching in G that saturates A. To show this, we prove that the Hall's Condition holds, i.e. for any $X \subseteq A, |N_F(X)| \geq |X|$. If $X = \emptyset$, the condition holds trivially. If $1 \leq |X| \leq n+1$, then the condition holds since for any $v \in X$

$$|N_F(X)| \geq |N_F(v)| = |L(v)| = n+1.$$

If $n+1 < |X| \leq 2(n+1)$, then X must contain two vertices u, w that are non-adjacent in G_{n+1}. By our assumption, $L(u) \cap L(w) = \emptyset$, thus

$$|N(u) \cup N(w)| \geq 2(k+1).$$

It follows that $|N(X)| \geq |N(u) \cup N(w)| \geq 2(k+1) \geq |X|$. $\square$

Alternative proof. We proceed similarly to the first proof until Case 2: If Case 2 holds, let $W \subseteq V(G_{n+1})$ be maximal such that $H = G_{n+1}[W]$ is L-list-colorable. If $H = G_{n+1}$ we are done, so assume for the sake of contradiction that H is a proper subgraph of G_{n+1}. Then there exists $v_0 \notin V(G_{n+1}) \setminus W$. Let φ be an L-list coloring of H with minimum $|S|$, where

$$S \coloneqq \{v \in W \mid \varphi(v) \in L(v_0)\}\,.$$

Furthermore, let $L(v_0) = \{c_1, \ldots, c_{n+1}\}$. As W is maximal, we know that for all $i \in [n+1]$ there exists a $v \in W$ such that $\varphi(v) = c_i$. Hence, $|S| \geq n+1$. By the pigeonhole principle, there must therefore be an $i \in [n+1]$ such that $\varphi(a_i), \varphi(b_i) \in L(v_0)$. As $L(a_i) \cap L(b_i) = \emptyset$, we have $|L(a_i) \mathbin{\dot\cup} L(b_i)| = 2(n+1)$.
Since $|W| < |V(G_{n+1})| = 2(n+1)$, there exists a color $c \in L(a_i) \mathbin{\dot\cup} L(b_i)$ that is not in the image of φ, in particular not in $L(v_0)$. W.lo.g. let $c \in L(b_i)$. Define a new coloring

$$\varphi'\colon W \to \bigcup_{v \in W} L(v), v \mapsto \begin{cases} c, & v = b_i \\ \varphi(v), & \text{otherwise.} \end{cases}$$

By the choice of c, this is also an L-list coloring of H. However, we see that

$$|\{v \in W \mid \varphi'(v) \in L(v_0)\}| = |S| - 1.$$

This contradicts the choice of φ. □

Long, constructive proof. We proceed similarly to the first proof until Case 2: Choose $x \in L(a_{n+1})$ and $y \in L(b_{n+1})$. Using x and y, define for all $v \in V(G_n)$

$$L'(v) \coloneqq \begin{cases} L(v) \setminus \{x, y\}\,, & |L(v) \cap \{x, y\}| \leq 1 \\ L(v) \setminus \{x\}\,, & \text{otherwise.} \end{cases}$$

Note that $|L'(v)| \geq n$ for all $v \in V(G_n)$ as we remove at most one element of each vertex' list. So, applying the induction hypothesis on $G_{n+1}[V(G_n)] \simeq G_n$, there exists a coloring c' with $c'(v) \in L'(v) \subseteq L(v)$ for all $v \in V(G_n)$. Note that each color is assigned at most once: The only vertex that a_i (b_i) is not adjacent to in G_n is b_i (a_i), so only they may share a color. But that can't be the case as $L(a_i) \cap L(b_i) = \emptyset$. Thus, each of the vertices in $V(G_n)$ is assigned a unique color and $|U| = 2n$ for $U \coloneqq \{c'(v)\colon v \in V(G_n)\}$. Keep in mind that because of $L(a_{n+1}) \cap L(b_{n+1}) = \emptyset$, we have $|L(a_{n+1}) \cup L(b_{n+1})| = 2(n+1)$. Therefore, $M \coloneqq (L(a_{n+1}) \cup L(b_{n+1})) \setminus U$ contains at least two elements. x must be one of those elements, as we have removed x from every list in L'. We distinguish two cases on $L(b_{n+1}) \cap M$.

Case 2.1: $L(b_{n+1}) \cap M \neq \emptyset$. Let $y' \in L(b_{n+1}) \cap M$. Then,

$$c\colon V(G_{n+1}) \to \mathbb{N}, v \mapsto \begin{cases} x, & v = a_{n+1} \\ y', & v = b_{n+1} \\ c'(v), & \text{otherwise} \end{cases}$$

defines a proper L-coloring as c' is a proper L'-coloring that doesn't use $x \in L(a_{n+1})$ and $y' \in L(b_{n+1})$.

Case 2.2: $L(b_{n+1}) \cap M = \emptyset$. Let the other color in M be $x' \neq x$. As $L(b_{n+1}) \cap M = \emptyset$, we must have $x' \in L(a_{n+1})$ and $y \in U$. Now, by our previous observation, there must be exactly one vertex $w \in V(G_n)$ such that $L'(w) \ni c'(w) = y$. But for $y \in L'(w)$, we must have that $\{x, y\} \subseteq L(w)$. Hence, define

$$c\colon V(G_{n+1}) \to \mathbb{N}, v \mapsto \begin{cases} x', & v = a_{n+1} \\ y, & v = b_{n+1} \\ x, & v = w \\ c'(v), & \text{otherwise.} \end{cases}$$

We have that $c(v) \in L(v)$ for all $v \in V(G_{n+1})$ as c' is a proper L'-coloring, $x' \in L(a_{n+1}), y \in L(b_{n+1})$ and $x \in L(w)$. It is also a proper coloring, as every vertex has a distinct color.

Thus, G_{n+1} is L-colorable and $\mathrm{ch}(G_{n+1}) = n + 1$ as L is arbitrary. □

Exercise 3 (Extra Credit Version)

Every directed graph without odd directed cycles has a kernel.[8]

Proof. Let D be a directed graph without odd directed cycles. W.l.o.g. is D non-empty. We will do an induction on the number of vertices $n = |V(D)|$. The claim is trivial for $n = 1$. So, consider $n > 1$.

Case 1: D is strongly connected. Let $v \in V(D)$ be an arbitrary vertex in D and let U be the set of vertices u such that there is an even directed u-v-walk. Clearly, as for every vertex $w \in V(D - U)$ there exists an odd w-v-walk W, for the vertex u following w in W there exists an even directed u-v-walk. So, for every $w \in V(D - U)$ there exists $u \in U$ such that $wu \in A(D)$.

Furthermore, U is independent: Assume not and $xy \in A(D)$ for $x, y \in U$. Let W_x and W_y be even directed walks from x to v and y to v respectively. As D is strongly connected, there exists a directed v-x-walk W. If W were odd, W_x and W would induce a closed directed walk of odd length that in particular would contain an odd directed cycle. ↯ So, W must be an even directed walk. But then tracing W, xy and then W_y would induce a closed directed walk of odd length again. ↯

Case 2: D is not strongly connected. Let C be a strongly connected component in D such that no vertex $v \in V(C)$ has an outgoing edge to a vertex outside of C. Note that C exists as the graph resulting from contracting all strongly connected components is a directed acyclic graph. The component corresponding to a sink of that resulting graph may then be chosen as C. Now, by applying Case 1 on C, we get a kernel $\hat{U}$ for C. Moreover, by applying the

[8] See page 133 in Diestel's *Graph Theory*.

induction hypothesis on the graph D' where we delete $\hat{U}$ and all predecessors of $\hat{U}$[9], we get a kernel U' for D'. Lastly, set $U := \hat{U} \cup U'$.

By construction, U is then a kernel of D: Indeed, U is independent, as by the choice of C there are no edges of the form uv with $u \in \hat{U}$ and $v \in U'$, and as D' contains all vertices of D that are neither in $\hat{U}$ nor are a predecessor of a vertex in $\hat{U}$, there are also no edges of the form vu with $u \in \hat{U}$ and $v \in U'$. Furthermore, every vertex $v \in V(D - U)$ has an outgoing edge to a vertex in U due to the fact that U' and U are kernels of their respective subgraphs of D.

This shows the claim. □

Remark. One could also proceed in the induction step in a slightly different way: As D is not strongly connected, there is a directed cut. If the corresponding vertex partition is $V(D) = A \dot\cup B$, where the edges from the cut go from A to B, we can first apply induction to get a kernel $\hat{U}$ of B, then remove the vertices from $\hat{U}$ and its predecessors and again find by induction a kernel U' of that graph. Then, $U = \hat{U} \dot\cup U'$ is again a kernel.
Lastly, the "standard task" was to show the existence of a kernel in a directed acyclic graph which can easily be done by iteratively adding the sinks to the resulting kernel, afterwards removing them together with their predecessors, and then iterating this until the graph is empty.

Exercise 4

(i) Every orientation of a bipartite graph has a kernel.

Proof. Let G be a bipartite graph. As every orientation D of G is a directed graph without odd directed cycles, it immediately follows from Exercise 3 that D has a kernel. □

(ii) $K_{2,4}$ is not 2-list-colorable.

Proof. Label the vertices such that the partition classes are $A = \{a_1, a_2\}$ and $B = \{b_i \mid i \in [4]\}$. Let $L(a_1) = \{1, 2\}, L(a_2) = \{3, 4\}$ and assign every list in $M := \{\{x, y\} \mid x \in \{1, 2\}, y \in \{3, 4\}\}$ to one $b \in B$. This is possible since $|M| = |\{1, 2\} \times \{3, 4\}| = 4$.
Observe that $K_{2,4}$ is not L-colorable: Suppose c were a proper L-coloring. Regardless of the choice of $i = c(a_1) \in \{1, 2\}$, there are two $j, k \in [4], j \neq k$, such that b_j and b_k have as lists $\{i, 3\}$ and $\{i, 4\}$ respectively. As $a_2b_j, a_2b_k \in E(K_{2,4})$, c could not be a proper L-coloring. ↯
So, as $K_{2,4}$ is not L-colorable, it is in particular not 2-list-colorable. □

[9] As in all vertices with an outgoing edge to a vertex in $\hat{U}$.

Sheet 5

Exercise 1

The complement of any bipartite graph is perfect.

Proof. Let G be a bipartite graph with parts X, Y. Since any induced subgraph of the complement of a bipartite graph is the complement of a bipartite graph (or empty graph), it suffices to show that $\chi(\overline{G}) = \omega(\overline{G})$. As $\chi(H) \geq \omega(H)$ for any graph H, we only need to show $\chi(\overline{G}) \leq \omega(\overline{G})$.
Recall Kőnig's Theorem: If G is bipartite, then the size of a smallest vertex cover of G equals the size of a largest matching in G. Let U be a smallest vertex cover of G and let $U_X = U \cap X$, $U_Y = U \cap Y$. There are no edges between $X - U_X$ and $Y - U_Y$. So, in $\overline{G}$ there is a clique of size

$$(|X| - |U_X|) + (|Y| - |U_Y|) = n - |U| \implies \omega(\overline{G}) \geq n - |U|\,,$$

where $n = |V(G)|$. By Kőnig's Theorem, there is a matching M of size $|U|$ in G.
We need to show that $\chi(\overline{G}) \leq n - |M|$: Indeed, if $e_1, \dots, e_{|M|}$ denote the edges of M, then we get a coloring of $\overline{G}$ by coloring the endpoints of e_i with i for $i = 1, \dots, |M|$ and coloring the remaining vertices with colors $\{|M| + 1, \dots, n - |M|\}$. Thus,

$$\chi(\overline{G}) \leq n - |M| = n - |U| \leq \omega(\overline{G}).$$ □

Alternative proof. Let G be a bipartite graph with parts X, Y. Again, it suffices to show that $\chi(\overline{G}) \leq \omega(\overline{G})$. As $\kappa(G) = \chi(\overline{G})$ and $\alpha(G) = \omega(\overline{G})$, where $\kappa(G)$ denotes the clique covering number and $\alpha(G)$ the independence number of G, this is equivalent to $\kappa(G) \leq \alpha(G)$.
So, consider a clique covering C of G. As $\omega(G) = 2$, C is of smallest size if and only if it contains a largest matching M of G, where the remaining vertices would be covered by K_1's. Thus, $\kappa(G) = n - |M|$ where $n = |V(G)|$. By Kőnig's Theorem, there is a vertex cover U of size $|M|$. It follows that $V \setminus U$ is an independent set. Indeed, otherwise there exists an edge that is not covered by U. Hence, $\kappa(G) \leq n - |M| \leq \alpha(G)$. □

Remark. The result also directly follows from the Weak Perfect Graph Theorem, but that's not the point.

Exercise 2

Remark. A graph is called a *comparability graph* if there exists a partial ordering of its vertex set such that two vertices are adjacent if and only if they are comparable.

Every comparability graph is perfect.

Constructive proof. Let G be a comparability graph, let $\leq$ denote the partial ordering on the vertex set and let $<$ denote the strict partial order dervied from $\leq$.

First, note that by simply restricting the partial order on the vertex set of the induced subgraph of G, we get a desired partial order on the induced subgraph. So, as induced subgraphs of comparability graphs are again comparability graphs, it suffices to show that $\chi(G) = \omega(G)$.
Enumerate the vertices $V = \{v_1, \dots, v_n\}$ in such a way that $v_i v_j \in E(G) \implies i < j$.[10] This is possible as the orientaion of G, where we orient $uv \in E(G)$ to $\vec{uv}$ if $u < v$, is a directed acyclic graph. Indeed, if it contained a directed cycle $w_1, \dots, w_m, w_1$, we would have by the transitivity of $\leq$ that we oriented $w_1 w_m$ to $w_1 \vec{w}_m$. $\lightning$
To generate an $\mathbb{N}$-coloring φ of G, we set $\varphi(v_j) = \max\{\varphi(v_i) \mid i < j, v_i < v_j\} + 1$ for $j = 1, \dots, n$, where $\max \emptyset \coloneqq 0$. Note that this is a proper coloring: If $v_i v_j \in E(G)$ with w.l.o.g. $i < j$, then by the definition of $\varphi(v_j)$, we must have $\varphi(v_j) > \varphi(v_i)$.
Now, let m denote the number of colors used by φ. There must then exist a vertex v_{i_m} with $\varphi(v_{i_m}) = m$. Again, by the construction of φ, there must exist a vertex $v_{i_{m-1}} < v_{i_m}$ such that $\varphi(v_{i_{m-1}}) = m - 1$. In general, if for $m \geq k > 1$ we have a vertex v_{i_k} with $\varphi(v_{i_k}) = k$, then by the construction of φ there must exist $i_{k-1} < i_k$ such that $v_{i_{k-1}} < v_{i_k}$ and $\varphi(v_{i_{k-1}}) = k - 1$.
Thus, we can construct a chain $C = v_{i_1} < \cdots < v_{i_m}$. And as all of them are pairwise comparable, C induces a clique of size m in G. As we always have $\chi(G) \geq \omega(G)$, we get

$$\chi(G) \leq m \leq \omega(G) \leq \chi(G) \implies \chi(G) = \omega(G). \qquad \square$$

Remark. In the language of partial orders, the statement is actually an equivalent formulation for *Mirsky's Theorem*. Funnily enough, someone in the problem class used the dual theorem to Mirsky's Theorem, *Dilworth's Theorem*.

Alternative proof. Let $G, \leq, <$ be as in the first proof and $H \subseteq G$ be an induced subgraph of G. As two vertices are adjacent if and only if they are comparable, we see that

- $\alpha(H)$ is the size of a greatest antichain (with respect to $\leq_H$),
- $\omega(H)$ is the size of a greatest chain (with respect to $\leq_H$).

Dilworth's Theorem now states that the minimum number m such that $V(H)$ can be partitioned into m chains is exactly the size of a greatest antichain, i.e. $\alpha(H)$.
So, as each chain is at most of size $\omega(H)$, we get

$$\alpha(H) = m \geq \frac{|V(H)|}{\omega(H)} \implies \alpha(H) \cdot \omega(H) \geq |V(H)|.$$

So, by Theorem 5.5.6 / Lovász's characterization for perfect graphs, G is perfect. $\square$

Elementary proof. As in the first proof, we know that it suffices to show that $\chi(G) = \omega(G)$. Fix the orientation on G given by the partial order.
We will now proceed by induction on $m = |E(G)|$: If $m = 0$, then G is empty and the statement is clear.

[10] In other words, we are topologically sorting the vertices.

Now, consider $m > 0$. Let S be the set of sources of G. By induction, $\chi(G - S) = \omega(G - S)$. Moreover, as the set of sources, S is independent. Furthermore, every maximum clique in G must contain one vertex (and due to the independence exactly one) of S, otherwise the clique could be enlarged. Thus, $\omega(G) = \omega(G-S)+1 = \chi(G-S)+1$. But at the same time, we can take a $\chi(G - S)$-coloring of $G - S$ and add S as the $(\omega(G - S) + 1)$-th color class. So, $\omega(G) \geq \chi(G)$, thus $\omega(G) = \chi(G)$. □

Exercise 3

The line graph $L(G)$ is perfect if and only if all odd cycles in G are triangles.

Proof. We first show the "if"-part by contraposition: Assume that G contains an odd cycle $C = v_1, \dots, v_k, v_1$ that is not a triangle. Then consider

$$H \coloneqq L(G)[\{v_1v_2, \dots, v_{k-1}v_k, v_kv_1\}].$$

As all the vertices $v_1, \dots, v_k$ are in particular pairwise distinct, we see that H is an induced odd cycle of length $k > 3$ in $L(G)$ where in particular $\chi(H) = 3 > \omega(H) = 2$. So, $L(G)$ is not perfect.
For the "only if"-direction, let all odd cycles in G be triangles. First note that induced subgraphs H of $L(G)$ are line graphs on the corresponding subgraph of G with $V(H)$ as edge set. However, as all odd cycles in subgraphs of G are also odd cycles in G itself, it suffices to show that $\chi(L(G)) = \omega(L(G))$. As the other inequality is trivial and $\chi'(G) = \chi(L(G))$, we would be done by showing $\chi'(G) \leq \omega(L(G))$.
To do this, note that $\Delta(G) \leq \omega(L(G))$ as the edges incident to a vertex with maximum degree induce a clique in $L(G)$. Let us first cover the case where G is 2-connected:
If G is bipartite, then we know by Kőnig's Line Coloring Theorem that

$$\chi'(G) = \Delta(G) \leq \omega(L(G)). \checkmark$$

Now, consider the case where G is not bipartite. Then G must contain a triangle $C = v_1v_2v_3v_1$. We will show that G is either K_4, or a union of $n \in \mathbb{N}$ "overlapping triangles" such that

$$V(G) = \{x, z, y_1, \dots, y_n\} \text{ and } E(G) = \{x, z\} \cup \{xy_i \mid i \in [n]\} \cup \{y_iz \mid i \in [n]\}.$$

As G is 2-connected, we can consider the (proper) ear decomposition $G_1, G_2, \dots, G_l$ of G starting with C[11]: If G has no ears, G would be the union of one "overlapping triangle". If G has ears, let P_1 be the "first ear" and let $\{v_j, v_k\} = V(P_1) \cap V(C)$. As v_j and v_k split C in an even and an odd segment of length 2 and 1 respectively, P_1 must be a path of length 2. Indeed, if it were an odd path (and thus of length 3 as C is a clique), we would get an odd cycle that is not a triangle. Similarly, if P_1 were an even path of length at least 4, then we would again get an odd cycle that is not a triangle. ↯

[11] Here, we let G_{i+1} be the graph resulting from adding an ear to G_i.

So, if G has ears, P_1 must be as above and G_2 would be a union of two overlapping triangles. From now on, we will hence use the notation as established for unions of overlapping triangles.
Now, assume that the ear decomposition contains a "second ear" P_2. If P_2 has endpoints y_1, y_2, then P_2 is of length 1, as otherwise P_2 would induce with either y_1zy_2 or y_1xzy_2 an odd cycle of length greather than 3. In that case, G_3 is isomorphic to K_4.
If P_2 has endpoints x, z, then by our argumentation above and using that $G_2[\{x, y_1, z\}]$ is a triangle, we see that P_2 must then be a path of length 2. So, G_3 would be the union of three overlapping edges.
Lastly, note that P_2 can't have one endpoint in $\{x, z\}$ and one endpoint in $\{y_1, y_2\}$. Indeed, w.l.o.g. consider the case where the endpoints are x, y_1. As $G_2[x, y_1, z]$ is a triangle, we get by our argumentation above that P_2 must be of length 2. But then $xP_2y_1zy_2x$ is an odd cycle of length 5. ↯
Now, in the case where $G_3 \simeq K_4$, we actually have $G = G_3$: Let $V(G_3) = \{a, b, c, d\}$. If P_3 were a "third ear" with endpoints a, b, then aP_3bcda is an odd cycle of length greater than 5 if P_3 is even. ↯ If P_3 is of odd length, then it must be at least of length 3, as $ab \in G_3$. But then aP_3bca is an odd cycle of length greater than 3. ↯
So, consider the case that G_3 is the union of 3 overlapping triangles. We will show that in this case G_k is the union of k overlapping triangles for all $3 \leq k \leq l$. We do so by doing induction on j with $k = 3$ as induction base. So, consider G_{k+1} and let P_k be the k-th ear.
As P_k generally must have endpoints in $\{x, y_i, y_j, z\}$ for some $i, j \in [k], i \neq j$, it follows directly that P_k could only be an x-z-path of length 2 as $G_k[\{x, y_i, y_j, z\}]$ is a union of two overlapping triangles. Thus, G_{k+1} the union of $k + 1$ overlapping triangles.
Now, in both cases we have that $\chi'(G) \leq \omega(L(G))$: As K_4 is an even clique, we have $\chi'(G) = 3 = \Delta(G) \leq \omega(L(G))$ if $G \simeq K_4$.
If G is the union of k overlapping triangles, $\Delta(G) = k + 1$. If $k = 1$, we have $\chi'(G) = 3 = \omega(L(G))$ as then $G \simeq K_3$ and $L(G) \simeq K_3$. For $k > 1$, we can color xz with $k + 1$, xy_i with i and y_iz with $i + 1$ for $i \in [k - 1]$, and xy_k with k and y_kz with 1.
This shows that $\chi'(G) \leq k + 1 = \Delta(G) \leq \omega(L(G))$.
Finally, for the general case we do an induction on $n = |V(G)|$: If $n \leq 2$, the claim is clear. So, consider $n > 3$. If G is disconnected, we can apply induction on each of the connected components $C^1, \ldots, C^m$ and would get

$$\chi'(G) = \max\left\{\chi'(C^i) \mid i \in [m]\right\} = \max\left\{\omega(L(C^j)) \mid i \in [m]\right\} = \omega(L(G)).$$

If G is 2-connected, the claim follows from the above.
So, let G have vertex-connectivity 1 and let v be a cutvertex and $G_1, G_2 \subseteq G$ be connected subgraphs formed by unions of blocks of G such that $V(G_1) \cap V(G_2) = \{v\}$. W.l.o.g. $\omega_1 := \omega(L(G_1)) \leq \omega(L(G_2)) =: \omega_2$. Applying induction, we get that there are edge colorings φ_1, φ_2 of G_1, G_2 using colors $[\omega_1], [\omega_2]$ respectively.
Permute the colors of φ_1 and φ_2 such that $\varphi_1(\{e \mid ve, e \in E(G_1)\}) = [\deg_{G_1}(v)]$ and

$$\varphi_2(\{e \mid ve, e \in E(G_2)\}) = \{\omega_2 - (\deg_{G_2}(v) - 1), \ldots, \omega_2\}.$$

If $\deg_{G_1}(v) + \deg_{G_2}(v) = \deg_G(v) \leq \omega_2$, so in particular $\omega(L(G)) = \omega_2$, then φ with $\varphi|_{E(G_i)} = \varphi_i$ for $i \in [2]$ defines an $\omega(L(G))$-edge coloring of G.
Otherwise, $\deg_G(v) = \omega(L(G))$. Consider

$$\varphi\colon E(G) \to [\omega(L(G))], e \mapsto \begin{cases} \varphi_1(e), & e \in E(G_1) \\ \varphi_2(e) + (\omega(L(G)) - \omega_2), & e \in E(G_2). \end{cases}$$

This defines an $\omega(L(G))$-edge coloring of G.
Hence, we get $\chi'(G) \leq \omega(L(G))$ in both cases. This completes the proof. $\square$

Alternative proof. By the arguments of above, it suffices to show that if G does not contain odd cycles of length at least 5, then $\chi(L(G)) = \omega(L(G))$.
To show that, assume the opposite and let $G = (V, E)$ be a counterexample for which $|E|$ is minimal. We must have $\Delta(G) \geq 3$. Indeed, if $\Delta(G) = 0$ then $L(G)$ would be empty and hence the equality would hold; if $\Delta(G) = 1$, then G contains disjoint K_2's and K_1's, so $L(G)$ would be an empty graph[12], hence $\chi(L(G)) = 1 = \omega(L(G))$; if $\Delta(G) = 2$, then G is the union of cycles (and contains at least one cycle) and paths, so – as G contains no odd cycles greater than 3 – $\chi(L(G)) = 2 = \Delta(G) = \omega(L(G))$ if G is bipartite and $\chi(L(G)) = 3 = \omega(L(G))$ otherwise.
So, $\Delta(G) \geq 3$. Let $e = uv \in E$ such that $\deg_G(v) < \Delta(G)$ unless G is regular. Then we choose e arbitrarily. As $G - e$ is not a counterexample, we can fix a proper edge coloring on $G - e$ with $\omega(L(G - e))$ many colors. Note that $k := \omega(L(G)) = \omega(L(G - e))$, as otherwise we could just extend the coloring to G by assigning e the "new" color. Also, $k \geq \Delta(G) \geq 3$. Now, in the coloring of $G - e$ there is a color "missing" at u and v respectively. If it were the same color, we could again extend the coloring to G. So, let the color "missing" at u be red and at v be blue.
We will now show that G is actually not regular: Assume otherwise. As we have $\Delta(G) \geq 3$, we know that $\Delta(G) = \omega(L(G))$. Indeed, as the edges incident to a vertex always induce a clique in $L(G)$, $\Delta(G) \leq \omega(L(G))$ holds. For the other direction, the case where $\omega(L(G)) = 3$ directly follows and for $\omega(L(G)) > 3$ we observe that a clique in $L(G)$ of size greater than 3 always corresponds to a star in G of the same size. Hence, the coloring on $G - e$ must use exactly $\Delta(G - e) = \Delta(G)$ colors, meaning that all vertices apart from u and v have no color missing while u and v have exactly one color missing. Therefore, as $\Delta(G) \geq 3$, there must be another color class, say green, that is different from red and blue, which induces a perfect matching in G. In particular, $|V(G)|$ must be even. However, we also know that the red color class induces a matching that misses exactly one vertex, namely u. That implies that $|V(G)|$ is odd. $\lightning$
So, we are in the case where $\deg_G(v) < \Delta(G)$. So, apart from blue, v must be missing another color, say green. Consider a longest alternating path P starting at v and an edge colored red at v that alternates between the colors red and green. By maximality, the last vertex must miss either red or green. If P doesn't end in u, we can just switch the colors on that path from red to green and green to red such that red is now also missing at v. So, we could extend the coloring to G in that case. $\lightning$

[12] In the sense that $E(L(G)) = \emptyset$.

So, P must end in u with a green edge.[13] In particular, P must be of even length. If P has length greater than 2, then together with uv it would form an odd cycle of length greater than 3 in G. ↯ So, P must have length two and the unique vertex $w \in V(P)$ in the interior of P is adjacent to u and v with uw colored green and wv colored red.
However, if we were to repeat this argument but for blue instead of green, we would get a contradiction as the w for blue, uniquely determined as the other endpoint of the one edge incident to v colored red, is the same w for green, meaning that uw needs to be colored blue and green at the same time. □

Remark. For the "only if"-direction, one could also use the Strong Perfect Graph Theorem: If $L(G)$ is not perfect, then $L(G)$ contains C_k or $\overline{C_k}$ for some odd $k > 3$. If $k = 5$, we have $C_5 \simeq \overline{C_5}$ and that C_5 would induce a C_5 in G. If $L(G)$ contains C_k for some odd $k > 5$, then, again, it would induce a C_k in G. Lastly, $L(G)$ actually can't contain a $\overline{C_k}$ for odd $k > 5$: Let $e_1, \ldots, e_k$ be the vertices of $\overline{C_k}$ such that e_i is not adjacent to e_{i+1}. Then e_1 and e_2 are not adjacent, thus they are independent edges in G. As $N_{\overline{C_k}}(e_1) \cap N_{\overline{C_k}}(e_2) \supseteq \{e_4, e_5, e_6\}$ and e_4, e_5 are not adjacent, we have that the edges e_1, e_4, e_2, e_5 (in that order) induce a C_4 in G. Now, e_6 needs to be an edge that is adjacent to e_1 and e_2 which is non-adjacent to e_4 and not equal to e_5. Clearly, such an edge can't exist. Hence, we actually covered all cases and are done. □

Exercise 4

Every bridgeless multigraph[14] G has a $\mathbb{Z}/k\mathbb{Z}$-flow for some integer k.

Proof. W.l.o.g. assume that G is connected, hence 2-edge connected. Again, it suffices to show that G has a k-flow for some $k \in \mathbb{N}$, or (dropping the parameter) a nowhere-zero $\mathbb{Z}$-flow. For the sake of contradiction, assume that G doesn't have such a $\mathbb{Z}$-flow. Let φ be a $\mathbb{Z}$-flow such that $|S(\varphi)|$ is minimum, where $S(\varphi)$ is the set of edges e such that $\varphi(e) = 0$. Note that φ exists as $\varphi \equiv 0$ is always a valid $\mathbb{Z}$-flow.
Now, consider $xy \in S$. As G is 2-edge connected, there is a y-x-path P in $G - xy$. In other words xyP forms a cycle C. Fix an orientation on C and let $k := \max\{|\varphi(e)| \mid e \in E(C)\}$. Let φ' be the $\mathbb{Z}$-flow that has flow value $k+1$ for the edges in the orientation of C, $-(k+1)$ for the edges in the opposite orientation of C, and assigns zero to every other edge. Clearly, this defines a $\mathbb{Z}$-flow. In particular, $\varphi + \varphi'$ is a $\mathbb{Z}$-flow that – by construction – assigns a non-zero value to each edge in C and each edge that had non-zero value in φ. So, as $e \in S(\varphi) \setminus S(\varphi + \varphi')$, $|S(\varphi + \varphi')| < |S(\varphi)|$. ↯ □

Remark. The 6-Flow Theorem shows that $k = 6$ can be chosen for all such multigraphs.

[13]Then, our argument above doesn't work as green would be missing at u after the switch.
[14]Note that by our convention multigraphs don't have loops.

Sheet 6

Exercise 1

For every graph with a spanning cycle the flow number is at most 4. This bound is tight.[15]

Proof. Let G be a graph with a spanning cycle C. By Corollary 6.3.2, it suffices to show that G has a $\mathbb{Z}_2^2$-flow. To construct such a flow φ, we proceed as follows:

- Let φ' be the $\mathbb{Z}_2$-circulation that assigns all (directed) edges of C the value 1 and all other edges 0.
- For each edge $e \in E(G) \setminus E(C)$, e closes two cycles with C. Fix for each such e one of those cycles C_e and let φ_e be the $\mathbb{Z}_2$-circulation that is only non-zero on C_e with value 1 for each directed edge in C_e. Then, $\varphi'' := \sum_{e \in E(G) \setminus E(C)} \varphi_e$ defines a $\mathbb{Z}_2$-circulation that is non-zero for each $e \in E(G) \setminus E(C)$.

With φ' and φ'' at hand, let $\varphi := (\varphi', \varphi'')$. This defines a $\mathbb{Z}_2^2$-flow.
Lastly, we show that the bound is tight: Consider K_4, which obviously has a spanning cycle. However, due to Proposition 6.4.2 and 6.4.5 (*ii*) K_4 has a flow number of 4. □

Exercise 2

There exists a graph G and a bridgeless subgraph $H \subseteq G$ such that G has a strictly smaller flow number than H.

Proof. Take $G = K_n$ for some $n > 4$ and $H = K_4$. As we have seen in the previous exercise, K_4 has a flow number of exactly 4. Furthermore, Proposition 6.4.1 and Proposition 6.4.3 state that the flow number of G is at most $3 < 4$. □

Remark. A harder question is asked in Problem 6.19 of Diestel's *Graph Theory*:

Find bridgeless graphs G and $H = G - e$ such that $2 < \varphi(G) < \varphi(H)$.

This problem is left as an exercise for the bored reader.

Exercise 3

The flow polynomial of K_4 is $p(x) = x^3 - 3x^2 + 2x$.

Proof. First, note that the flow polynomial is invariant under subdivisions. Indeed, if H is a group, e is an edge of a graph G and G' is the resulting graph from G by subdividing e, then due to the flow conservation at the new vertex of G', it is apparent that the (nowhere-zero) H-flows of G are in a one-to-one correspondence to the H-flows of G'. As the flow polynomial is by Theorem 6.3.1 constructed to count exactly that quantity, we have that G and G' have the same flow polynomial.

[15] In other words, the flow number of a Hamiltonian graph is at most 4.

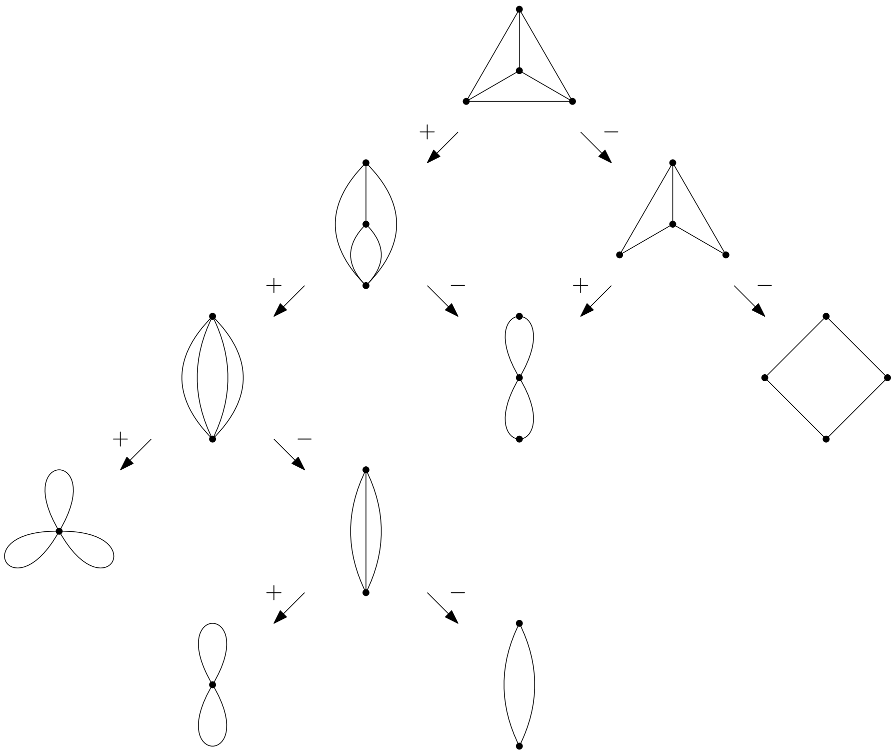

Figure 3: Scheme for the deletion-contraction formula

In particular, the flow polynomial of the graph with exactly two parallel edges and the flow polynomial of cycles is the same as the flow polynomial of a loop, which is given by x. In general, we know from Theorem 6.3.1 that if G has exactly m loops as edges, its flow polynomial is x^m.
Finally, by applying the deletion-contraction formula established in Theorem 6.3.1, where we proceed as in Figure 3, we see that the flow polynomial of K_4 is

$$p(x) = x^3 - x^2 + x - 2x^2 + x = x^3 - 3x^2 + 2x. \qquad \square$$

Remark. Just for fun, I wrote some Haskell code to get the flow polynomial for a general graph, which is represented by a list of its edges.

```
replace :: Eq a => a -> a -> a -> a
replace a b x = if x == a then b else x

mapTuple :: (a -> b) -> (a, a) -> (b, b)
mapTuple f (a1, a2) = (f a1, f a2)

contract :: Eq a => (a, a) -> [(a, a)] -> [(a, a)]
contract p ps = map (mapTuple $ replace (fst p) (snd p)) ps
```

```
tutte m []          = 1:(take (m-1) [0,0..])
tutte m (p:ps)
| (fst p == snd p) = 0:(tutte (m-1) ps)
| otherwise        = zipWith (-) (tutte m $ contract p ps) (tutte m ps)

tutte_init l = tutte (1 + length l) l

clique 0 = []
clique 1 = []
clique n = clique (n-1) ++ [(i, n) | i <- [1..(n-1)] ]

biclique m n = [(i, j + m) | i <- [1..m], j <- [1..n]]

main = putStrLn (show $ tutte_init $ clique 4)
```

Exercise 4

A plane triangulation is 3-colorable if and only if all its vertices have even degree.

Proof. Let G be a plane triangulation. Clearly, as K_3 is trivially 3-colorable and also 2-regular, the statement is true for $G = K_3$. So, consider the case $|V(G)| > 3$. It is a known fact that G is then 3-connected. In particular, G^*, the dual of G, is a proper plane graph that is cubic since G is a plane triangulation. Indeed, G^* has no loops as G has no bridges. Furthermore, parallel edges in G^* would imply that two faces share at least two edges in G which would then form a cut. However, G has no such cuts as it is 3-connected. It is also not hard to see that G^* is 2-connected.[16] Indeed, if f is a face in G, then the faces that share with f a vertex induce a cycle in G^* according to the clockwise order of the faces. So, in G^*, removing f would not disconnect the graph. Now, as $\chi(G) \geq \omega(G) \geq 3$, we know by Theorem 6.5.3 that G is 3-colorable if and only if G^* has flow number 3. By Proposition 6.4.2, we know that G^* has flow number 3 if and only if it is bipartite. So, it suffices to show that G^* is bipartite if and only if all of G's vertices have even degree.
This is also not too hard to see: For now, we will call G even if all of its vertices have even degree. As G's vertices are in one-to-one correspondence to the faces of G^*, we see that G is even if and only if every face in G^* is bounded by an even cycle. By a previous exercise[17], we know that G^* is bipartite if and only if every face is bounded by an even cycle. This completes the proof. □

Remark. One can also show directly the "if"-direction of the statement: If $v \in V(G)$ is a vertex of odd degree, then the neighboring vertices around v must form an odd cycle due to G being a plane triangulation and would therefore form an odd wheel together with v. Hence, as we know that odd wheels have chromatic number 4, G is not 3-colorable.

[16]Note that the dual of any plane graph is connected.
[17]See Exercise 3 on Sheet 3 or Problem 24 in Chapter 4 of Diestel's *Graph Theory.*

Sheet 7

Exercise 1

Our definition of regular pair in class is equivalent to the one given in Diestel's *Graph Theory*.

Proof. We first go from our definition to the one given in the textbook: Let (X, Y) be an (ε, d)-regular pair according to our definition in class. For one, we get

$$|e_G(X,Y) - d\,|X|\,|Y|| \leq \varepsilon\,|X|\,|Y| \implies |d(X,Y) - d| \leq \varepsilon.$$

Now, consider $X' \subseteq X, Y' \subseteq Y$ with $|X'| \geq \gamma\,|X|\,, |Y| \geq \gamma\,|Y|$ for some $1 \geq \gamma > 0$.
It follows that

$$\begin{aligned} |d(X',Y') - d(X,Y)| \leq |d(X',Y') - d| + |d - d(X,Y)| &\leq \varepsilon \frac{|X|\,|Y|}{|X'|\,|Y'|} + \varepsilon \\ \leq \varepsilon \left(\frac{1+\gamma^2}{\gamma^2}\right) &\leq \frac{2\varepsilon}{\gamma^2}, \end{aligned}$$

where we applied the regularity property and triangle inequality. Now, to go to regularity as defined in the textbook, we set

$$\frac{2\varepsilon}{\gamma^2} = \gamma \implies \gamma = \sqrt[3]{2\varepsilon}.$$

So, we get that (X, Y) is a $\delta = \sqrt[3]{2\varepsilon}$-regular pair as by the definition given in the textbook.
For the other direction, let (X, Y) be a δ-regular pair for some $0 < \delta \leq 1$ according to the definition in the textbook. Then we get for $d := d(X, Y)$ that

$$|e_G(X',Y') - d\,|X'|\,|Y'|| = |X'|\,|Y'|\,|d(X',Y') - d| \leq \delta\,|X'|\,|Y'| \leq \delta\,|X|\,|Y|$$

for $X' \subseteq X, Y' \subseteq Y$ with $|X'| \geq \delta\,|X|\,, |Y'| \geq \delta\,|Y|$. If $|X'| < \delta\,|X|$, then

$$|e_G(X',Y') - d\,|X'|\,|Y'|| \leq \max\{\delta\,|X|\,|Y'|\,, d \cdot \delta\,|X|\,|Y'|\} \leq \delta\,|X|\,|Y|\,.$$

One can arrive at the same inequality for $|Y'| < \delta\,|Y|$ by symmetry. So, (X, Y) is a (δ, d)-regular pair as according to our definition. Hence, both definitions are equivalent. □

Exercise 2

The Triangle Counting Lemma is not true if only one of its three parts is regular.

Proof. Let $n \in \mathbb{N}$ be even. Consider the tripartite graph G with $V(G) = V_1 \cupdot V_2 \cupdot V_3$ with $V_i = \{v_i^1, \ldots, v_i^n\}$ and

$$E(G) = \left\{v_1^i v_2^j \mid i,j \in [n]\right\} \cup \left\{v_1^i v_3^j \mid i \in [n], j \in \left[\frac{n}{2}\right]\right\} \cup \left\{v_2^i v_3^j \mid i \in [n], \frac{n}{2} + 1 \leq j \leq n\right\}.$$

Note that
$$d_{1,2} = 1, d_{1,3} = d_{2,3} = \frac{n \cdot \frac{n}{2}}{n^2} = \frac{1}{2}$$
and that (V_1, V_2) is $(\varepsilon, d_{1,2})$-regular for every $\varepsilon > 0$ as
$$e_G(X,Y) = |X|\,|Y| \implies |e_G(X,Y) - |X|\,|Y|| = 0 < \varepsilon.$$
Furthermore, due to the adjacencies of the vertices in V_3, G is triangle-free. On the other hand,
$$d_{1,2}d_{1,3}d_{2,3}\,|V_1|\,|V_2|\,|V_3| = \frac{n^3}{4}.$$
□

The Triangle Counting Lemma is true if two of its three parts are regular.

Proof. We mimick the proof of the General Counting Lemma: Let $G = (V_1 \cup V_2 \cup V_3, E)$ be tripartite such that $(V_1, V_2), (V_2, V_3)$ are $(\varepsilon, d_{1,2})$-regular and $(\varepsilon, d_{2,3})$-regular for some $d_{1,2}, d_{2_3} \in [0, 1]$. Set
$$d_{1,3} = d(V_1, V_3) = \frac{e(V_1, V_3)}{|V_1|\,|V_3|},$$
where we may assume that $|V_1| > 0 < |V_3|$ as otherwise G would be triangle-free (and not properly tripartite). We will then show that the number of triangles in G is in the interval $(d_{1,2}d_{2,3}d_{1,3} \pm 2\varepsilon)\,|V_1|\,|V_2|\,|V_3|$. In other words, choosing $\varepsilon = \gamma/2$ will suffice.
To make everything concrete, let $V(C_3) = \{1, 2, 3\}$. We will proceed iteratively: Let Φ_1 be the set of partite homomorphisms φ from $C_3 - \{1, 2\} - \{2, 3\}$ to G such that $\varphi(i) \in V_i$ for all $i \in [3]$. Clearly, we have $|V_2|$ choices for $\varphi(2)$ (as its just an isolated vertex in $C_3 - \{1, 2\} - \{2, 3\}$) and $\varphi(1) = a \in V_1, \varphi(2) = c \in V_3$ is possible if and only if $ac \in E$. Therefore,
$$|\Phi_1| = e(V_1, V_3) \cdot |V_2| = d_{1,3}\,|V_1|\,|V_2|\,|V_3|\,.$$
Next, let Φ_2 be the set of partite homomorphisms φ from $C_3 - \{2, 3\}$ to G such that $\varphi(i) \in V_i$ for all $i \in [3]$. Clearly, we have
$$\begin{aligned}
|\Phi_2| &= \sum_{\varphi \in \Phi_1} \mathbb{1}_E(\varphi(1), \varphi(2)) \\
&= \sum_{\varphi \in \Phi_1} (\mathbb{1}_E(\varphi(1), \varphi(2)) - d_{1,2} + d_{1,2}) \\
&= d_{1,2}\,|\Phi_1| + \sum_{\varphi \in \Phi_1} (\mathbb{1}_E(\varphi(1), \varphi(2)) - d_{1,2})\,.
\end{aligned}$$
Now, since $d_{1,2}\,|\Phi_1| = d_{1,2}d_{1,3}\,|V_1|\,|V_2|\,|V_3|$ as desired, we focus on bounding the latter term. Clearly,
$$\begin{aligned}
&\sum_{\varphi \in \Phi_1} (\mathbb{1}_E(\varphi(1), \varphi(2)) - d_{1,2}) \\
= &\sum_{v_3 \in V_3} \sum_{v_1 \in V_1 \cap N(v_3), v_2 \in V_2} (\mathbb{1}_E(\varphi(1), \varphi(2)) - d_{1,2})
\end{aligned}$$

$$\begin{aligned}
&= \sum_{v_3\in V_3} \left(\left(\sum_{v_1\in V_1\cap N(v_3), v_2\in V_2} \mathbb{1}_E(\varphi(1),\varphi(2)) \right) - \left(\sum_{v_1\in V_1\cap N(v_3), v_2\in V_2} d_{1,2} \right) \right) \\
&= \sum_{v_3\in V_3} \left(e(V_1\cap N(v_3), V_2) - d_{1,2}\,|V_1\cap N(v_3)|\,|V_2| \right),
\end{aligned}$$

so the triangle inequality and ε-regularity of (A,B) imply

$$\begin{aligned}
\left| \sum_{\varphi\in\Phi_1} \left(\mathbb{1}_E(\varphi(1),\varphi(2)) - d_{1,2} \right) \right| &\le \sum_{v_3\in V_3} \left| e(V_1\cap N(v_3), V_2) - d_{1,2}\,|V_1\cap N(v_3)|\,|V_2| \right| \\
&\le \sum_{v_3\in V_3} \varepsilon\,|V_1|\,|V_2| \\
&= \varepsilon\,|V_1|\,|V_2|\,|V_3|\,.
\end{aligned}$$

Thus, $|\Phi_2| \in (d_{1,2}d_{1,3} \pm \varepsilon)\,|V_1|\,|V_2|\,|V_3|$.
Lastly, let Φ_3 be the set of partite homomorphisms φ from C_3 to G such that $\varphi(i) \in V_i$ for all $i \in [3]$. Similarly to before, we see that

$$\begin{aligned}
|\Phi_3| &= \sum_{\varphi\in\Phi_2} \mathbb{1}_E(\varphi(2),\varphi(3)) \\
&= \sum_{\varphi\in\Phi_2} \left(\mathbb{1}_E(\varphi(2),\varphi(3)) - d_{2,3} + d_{2,3} \right) \\
&= d_{2,3}\,|\Phi_2| + \sum_{\varphi\in\Phi_2} \left(\mathbb{1}_E(\varphi(2),\varphi(3)) - d_{2,3} \right).
\end{aligned}$$

Now, since $d_{2,3}\,|\Phi_2| \in (d_{1,2}d_{1,3}d_{2,3} \pm \varepsilon)\,|V_1|\,|V_2|\,|V_3|$ as desired, we focus on bounding the latter term. Clearly,

$$\begin{aligned}
&\sum_{\varphi\in\Phi_2} \left(\mathbb{1}_E(\varphi(2),\varphi(3)) - d_{2,3} \right) \\
&= \sum_{v_1\in V_1} \sum_{v_2\in V_2\cap N(v_1), v_3\in V_3\cap N(v_1)} \left(\mathbb{1}_E(\varphi(2),\varphi(3)) - d_{2,3} \right) \\
&= \sum_{v_1\in V_1} \left(\left(\sum_{\substack{v_2\in V_2\cap N(v_1),\\ v_3\in V_3\cap N(v_1)}} \mathbb{1}_E(\varphi(2),\varphi(3)) \right) - \left(\sum_{\substack{v_2\in V_2\cap N(v_1),\\ v_3\in V_3\cap N(v_1)}} d_{2,3} \right) \right) \\
&= \sum_{v_1\in V_1} \left(e(V_2\cap N(v_1), V_3\cap N(v_1)) - d_{2,3}\,|V_2\cap N(v_1)|\,|V_3\cap N(v_1)| \right),
\end{aligned}$$

so the triangle inequality and ε-regularity of (B,C) imply

$$\left| \sum_{\varphi\in\Phi_2} \left(\mathbb{1}_E(\varphi(2),\varphi(3)) - d_{2,3} \right) \right|$$

$$
\begin{aligned}
&\leq \sum_{v_1 \in V_1} \left| e(V_2 \cap N(v_1), V_3 \cap N(v_1)) - d_{2,3} \left| V_2 \cap N(v_1) \right| \left| V_3 \cap N(v_1) \right| \right| \\
&\leq \sum_{v_1 \in V_1} \varepsilon \left| V_2 \right| \left| V_3 \right| \\
&= \varepsilon \left| V_1 \right| \left| V_2 \right| \left| V_3 \right| .
\end{aligned}
$$

Thus, $|\Phi_3| \in (d_{1,2} d_{1,3} d_{2,3} \pm 2\varepsilon) |V_1| |V_2| |V_3|$ as desired. □

Remark. Asking Professor Schacht, it became clear what kind of relaxation is allowed in the General Counting Lemma: The General Counting Lemma goes through if none of the non-regular pairs meet. Note that this is exactly what we used to construct a counterexample when two of the three pairs are non-regular in G. If none of the non-regular pairs meet, i.e. the pairs that are non-regular correspond to a matching M of our graph F whose number of partite homomorphisms to G we want to count, then in the induction base we take exactly those pairs and get the desired number without error.

Exercise 3

The number of partite homomorphisms of C_4 in a bipartite graph $G = (X \cup Y, E)$ with $|E| = d\,|X|\,|Y|$ is at least $d^4 |X|^2 |Y|^2$.

Remark. To be non-ambiguous, we try to count the number of closed walks $xyx'y'x$ in G such that $x, x' \in X$ and $y, y' \in Y$.

Proof. Recall that by the Cauchy-Schwarz inequality, we have

$$\langle x, y \rangle \leq \|x\|_2 \, \|y\|_2$$

for all $x, y \in \mathbb{R}^n, n \in \mathbb{N}$. In particular, if we set $x = (a_1/n, \ldots, a_n/n)$ and $y = (1, \ldots, 1)$ for any $a_1, \ldots, a_n \in \mathbb{R}$, we get

$$\frac{1}{n} \sum_{i=1}^{n} a_i \leq \sqrt{\sum_{i=1}^{n} \frac{a_i^2}{n^2}} \cdot \sqrt{1^2 + \cdots + 1^2} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} a_i^2} \tag{$*$}$$

Now, to count the number of partite homomorphisms, let $d(u, v) = |N(u) \cap N(v)|$ for $u, v \in V(G)$. We proceed as follows: For fixed $(y, y') \in Y^2$, the number of such homomorphisms containing y and y' in that order is given by $d(y, y')^2$ as this is exactly the number of choices we have for x and x'.
So, the number of homomorphisms N is given by

$$
\begin{aligned}
N &= \sum_{(y,y') \in Y^2} d(y, y')^2 \\
&= |Y|^2 \left(\sum_{(y,y') \in Y^2} \frac{d(y, y')^2}{|Y|^2} \right)
\end{aligned}
$$

$$\overset{(*)}{\geq} |Y|^2 \left(\frac{1}{|Y|^2} \sum_{(y,y')\in Y^2} d(y,y') \right)^2 .$$

Observe that the inner sum counts the number of triples $(x, y, y') \in X \times Y^2$ such that $x \in N(y) \cap N(y')$. Counting from the perspective of the vertices in X, that quantity is

$$\begin{aligned}
\sum_{x\in X} \deg(x)^2 &= |X| \left(\frac{1}{|X|} \sum_{x\in X} \deg(x)^2 \right) \\
&\overset{(*)}{\geq} |X| \left(\frac{1}{|X|} \sum_{x\in X} \deg(x) \right)^2 \\
&= |X| \left(\frac{|E|}{|X|} \right)^2 \\
&= d^2 |X|\, |Y|^2 .
\end{aligned}$$

Plugging that in, we finally get

$$N \geq |Y|^2 \left(\frac{d^2\, |X|\, |Y|^2}{|Y|^2} \right)^2 = d^4\, |X|^2\, |Y|^2 . \qquad \square$$

Exercise 4

Let (X, Y) be an (ε, d)-regular pair with $d^3 > \varepsilon > 0$ and let M be a largest matching in such a pair.

1. There are less than $\sqrt[3]{\varepsilon}\, |X|$ many vertices unmatched in X or $\sqrt[3]{\varepsilon}\, |Y|$ many vertices unmatched in Y.
2. If $m := |X| = |Y|$ and the minimum degree is at least dm, then (X, Y) contains a perfect matching.

Proof.

1. Assume otherwise. Let $X' \subseteq X$ and $Y' \subseteq Y$ be the set of unmatched vertices in X and Y respectively. By assumption, $|X'| \geq \sqrt[3]{\varepsilon}|X|$ and $|Y'| \geq \sqrt[3]{\varepsilon}|Y|$. Note that $E(X', Y') = \emptyset$. Otherwise, there would be an edge e with two unmatched endpoints, making $M \cup \{e\}$ a larger matching. ↯ However, as (X, Y) is (ε, d)-regular, we get

   $$d\varepsilon^{\frac{2}{3}}\, |X|\, |Y| \leq \left| e(X', Y') - d\, |X'|\, |Y'| \right| \leq \varepsilon\, |X|\, |Y| ,$$

   which implies $d \leq \sqrt[3]{\varepsilon}$, a contradiction. $\square$
2. Assume otherwise. Then M is imperfect. As M is of maximum size, there is no M-alternating path connecting unmatched vertices. Furthermore, as $|X| = |Y|$, there is at least one unmatched vertex x and y in X and Y respectively. Due to the minimum degree condition $Y' := N(x)$ and $X' := N(y)$ have at least size dm. Furthermore, all of the vertices in X' and Y' are covered by M as otherwise there

would be an improving M-alternating path. Similarly, $e(X', Y') = \emptyset$ as otherwise we would get an M-alternating path connecting x and y. However, as (X, Y) is (ε, d)-regular, we get

$$d^3 m^2 \leq |e(X', Y') - d\,|X'|\,|Y'|| \leq \varepsilon m^2 \implies d^3 \leq \varepsilon. \lightning \qquad \square$$

Sheet 8

Exercise 1

Let $\varepsilon > 0$ and let $G = (V_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} V_l, E(G))$ be an l-partite graph and F be a graph with $V(F) = [l]$. If every bipartite pair (V_i, V_j) is $(\varepsilon, d_{i,j})$-regular for some $d_{i,j} \geq 0$, then

$$\left| |\mathrm{Hom}_{\mathrm{ind}}(F, G)| - \prod_{ij \in E(F)} d_{i,j} \cdot \prod_{ij \notin E(F)} (1 - d_{i,j}) \cdot \prod_{k=1}^{l} |V_k| \right| \leq \varepsilon \binom{l}{2} \prod_{k=1}^{l} |V_k|\,,$$

where $\mathrm{Hom}_{\mathrm{ind}}(F, G)$ is the set of induced partite homomorphisms $\varphi\colon V(F) \to V(G)$ such that $\varphi(i) \in V_i$ for all $i \in [l]$, and $ij \in E(F)$ if and only if $\varphi(i)\varphi(j) \in E(G)$.

Proof. Let H be the l-partite graph with $V(H) = V(G)$ and for all $1 \leq i < j \leq l$

$$E_H(V_i, V_j) = \begin{cases} E_G(V_i, V_j), & ij \in E(F) \\ \{uv \mid u \in V_i, v \in V_j\} \setminus E_G(V_i, V_j), & ij \notin E(F). \end{cases}$$

By construction, $\mathrm{Hom}_{\mathrm{ind}}(F, G) = \mathrm{Hom}(K_l, G)$ where we set $V(K_l) = [l]$. Furthermore, note that (V_i, V_j) is $(\varepsilon, d_{i,j})$-regular with respect to H for $ij \in E(F)$ as the same is the case for G. For $ij \notin E(F)$, (V_i, V_j) is $(\varepsilon, 1 - d_{i,j})$-regular with respect to H as

$$\begin{aligned} |e_H(U, V) - (1 - d_{i,j})\,|U|\,|V|| &= |(|U|\,|V| - e_G(U, V)) - (1 - d_{i,j})\,|U|\,|V|| \\ &= |d_{i,j}\,|U|\,|V| - e_G(U, V)| \\ &\leq \varepsilon\,|V_i|\,|V_j| \end{aligned}$$

holds for all $U \subseteq V_i, V \subseteq V_j$ as (V_i, V_j) is $(\varepsilon, d_{i,j})$-regular with respect to G.
The claim now follows from the usual General Counting Lemma. $\square$

Exercise 2

Remark. For a bipartite graph $G = (X \mathbin{\dot\cup} Y, E)$ and $d \in [0, 1]$ we define

$$\mathrm{dev}(G, d) = \sum_{x_0, x_1 \in X} \sum_{y_0, y_1 \in Y} \prod_{i,j \in \{0,1\}} (\mathbb{1}_E(x_i, y_j) - d)\,.$$

If $\mathrm{dev}(G, d) \leq \varepsilon\,|X|^2\,|Y|^2$, then (X, Y) is $(\sqrt[4]{\varepsilon}, d)$-regular.

Proof. We show the assertion through a direct calculation: Let $X' \subseteq X$ and $Y' \subseteq Y$. It follows with the Cauchy-Schwarz inequality and $\operatorname{dev}(G,d) \leq \varepsilon\,|X|^2\,|Y|^2$ that

$$
\begin{aligned}
&\left(e(X',Y') - d\,|X'|\,|Y'|\right)^4 \\
&= \left(\sum_{x\in X}\sum_{y\in Y} \mathbb{1}_{X'}(x)\mathbb{1}_{Y'}(y)\left(\mathbb{1}_E(x,y)-d\right)\right)^4 \\
&= \left(\sum_{x\in X} \mathbb{1}_{X'}(x)\sum_{y\in Y} \mathbb{1}_{Y'}(y)\left(\mathbb{1}_E(x,y)-d\right)\right)^4 \\
&\leq \left(\sqrt{\sum_{x\in X}\mathbb{1}_{X'}(x)^2}\sqrt{\sum_{x\in X}\left(\sum_{y\in Y}\mathbb{1}_{Y'}(y)\left(\mathbb{1}_E(x,y)-d\right)\right)^2}\right)^4 \\
&= |X'|^2\left(\sum_{x\in X}\left(\sum_{y\in Y}\mathbb{1}_{Y'}(y)\left(\mathbb{1}_E(x,y)-d\right)\right)^2\right)^2 \\
&= |X'|^2\left(\sum_{x\in X}\sum_{y_0,y_1\in Y}\mathbb{1}_{Y'}(y_0)\mathbb{1}_{Y'}(y_1)\left(\mathbb{1}_E(x,y_0)-d\right)\left(\mathbb{1}_E(x,y_1)-d\right)\right)^2 \\
&= |X'|^2\left(\sum_{y_0,y_1\in Y}\mathbb{1}_{Y'}(y_0)\mathbb{1}_{Y'}(y_1)\sum_{x\in X}\left(\mathbb{1}_E(x,y_0)-d\right)\left(\mathbb{1}_E(x,y_1)-d\right)\right)^2 \\
&\leq |X'|^2\left(\sqrt{\sum_{y_0,y_1\in Y}\mathbb{1}_{Y'}(y_0)^2\mathbb{1}_{Y'}(y_1)^2}\sqrt{\sum_{y_0,y_1\in Y}\left(\sum_{x\in X}\left(\mathbb{1}_E(x,y_0)-d\right)\left(\mathbb{1}_E(x,y_1)-d\right)\right)^2}\right)^2 \\
&= |X'|^2\,|Y'|^2\sum_{y_0,y_1\in Y}\left(\sum_{x\in X}\left(\mathbb{1}_E(x,y_0)-d\right)\left(\mathbb{1}_E(x,y_1)-d\right)\right)^2 \\
&= |X'|^2\,|Y'|^2\operatorname{dev}(G,d) \\
&\leq \varepsilon\,|X|^4\,|Y|^4 .
\end{aligned}
$$

Taking the fourth root, we get

$$\left|e(X',Y') - d\,|X'|\,|Y'|\right| \leq \sqrt[4]{\varepsilon}\,|X|\,|Y|\,.$$

Thus, (X,Y) is $(\sqrt[4]{\varepsilon}, d)$-regular. □

Exercise 3

For every $\gamma > 0$ there exists $\delta > 0$ such that for sufficiently large n every n-vertex graph with more than $((l-2)/(l-1)+\gamma)\,n^2/2$ edges contains not only one, but even at least δn^l copies of K_l.

Proof. Given $\gamma > 0$ set $d_0 = \gamma/4, t_0 = \max\{l, \lceil 4/\gamma \rceil\}$, and

$$\varepsilon = \min\left\{\frac{1}{2}, \frac{\gamma}{8}, \frac{(\frac{\gamma}{4})^{\binom{l}{2}}}{\binom{l}{2}+1}\right\}.$$

Choose $T_0 = T_0(\varepsilon, t_0)$ as according to the Regularity Lemma and set $n_0 \geq T_0$

$$\delta = \frac{(\frac{\gamma}{4})^{\binom{l}{2}}}{(2T_0)^l\left(\binom{l}{2}+1\right)}.$$

Given G on $n \geq n_0$ vertices with more than $((l-2)/(l-1)+\gamma)\, n^2/2$ edges, apply the Regularity Lemma with ε and t_0 to obtain a partition $V_0 \cup V_1 \cup \ldots \cup V_t = V(G)$, where $t_0 \leq t \leq T_0$. As usual, we remove $xy \in E(G)$ if

1. $\{x, y\} \cap V_0 \neq \emptyset$,
2. $\{x, y\} \subseteq V_i$ for some $i \in [t]$,
3. $xy \in E(V_i, V_j)$ and (V_i, V_j) is not an ε-regular pair,
4. $xy \in E(V_i, V_j)$ and $d(V_i, V_j) < d_0$.

We get that there are at most εn^2 edges of Type 1, at most

$$t\binom{\frac{n}{t}}{2} \leq \frac{n^2}{2t} \leq \frac{n^2}{2t_0}$$

edges of Type 2, at most

$$\varepsilon t^2\left(\frac{n}{t}\right)^2 = \varepsilon n^2$$

edges of Type 3, and at most

$$\binom{t}{2} d_0 \left(\frac{n}{t}\right)^2 \leq \frac{d_0}{2} \cdot n^2$$

edges of Type 4. Therefore, we have removed at most

$$\left(\frac{1}{2t_0} + 2\varepsilon + \frac{d_0}{2}\right) n^2 \leq \gamma \frac{n^2}{2}$$

edges. So, as there are still more than $(l-2)/(l-1)n^2/2$ edges left, Turán's Theorem implies that there exists a K_l in the remaining graph. By our choice of which edges got removed, this implies that w.l.o.g. (V_i, V_j) is ε-regular with density at least d_0 for all $1 \leq i < j \leq l$. Hence, the Counting Lemma implies that there are at least

$$\left[d_0^{\binom{l}{2}} - \varepsilon\binom{l}{2}\right]\left(\frac{(1-\varepsilon)n}{t}\right)^l \geq \frac{d_0^{\binom{l}{2}} - \varepsilon\binom{l}{2}}{(2T_0)^l} \cdot n^l \geq \delta n^l$$

copies of K_l as desired. $\square$

Exercise 4

Deduce the Erdős-Stone theorem (Theorem 7.1.2) from Turán's theorem.

Solution. We will use the Embedding Lemma. Recall that the Embedding Lemma states:

Proposition (Embedding Lemma)**.** For all $\Delta \in \mathbb{N}, k \in \mathbb{N}, d_0 > 0$ there exists $\varepsilon > 0, M \in \mathbb{N}$ such that: If

- $G = (V_1 \uplus \ldots \uplus V_k, E(G))$ is a graph where (V_i, V_j) is $(\varepsilon, d_{i,j})$-regular with $d_{i,j} \geq d_0$ for all $1 \leq i < j \leq l$, and
- $H = (U_1 \uplus \ldots \uplus U_k, E(H))$ with $\Delta(H) \leq \Delta$ and $|V_i| \geq M\,|U_i|$ for all $i \in [k]$,

then $H \subseteq G$.

So, let $r \geq 2, s \geq 1$, and $\gamma > 0$ where w.l.o.g. $\gamma < 1$. Set $\Delta = \Delta(K_s^r) = (r-1)s, k = r$ and $d_0 = \gamma/2$. From the Embedding Lemma, we get $\varepsilon' > 0$ and $M \in \mathbb{N}$. Set $\varepsilon = \min\{\varepsilon', 1/2, \gamma/8\}$. Given ε and $t_0 = \max\{k, \lceil 2/\gamma \rceil\}$, we get from the Regularity Lemma $T_0 \in \mathbb{N}$. Lastly, let $n_0 = \max\{2M \cdot T_0 \cdot s, 2/\sqrt{\gamma}\}$.
So, let G be a graph with $n \geq n_0$ vertices and at least $t_{r-1}(n) + \gamma n^2$ edges. From the Regularity Lemma, we get a partition $V = V_0 \uplus V_1 \uplus \ldots \uplus V_t$ where $t_0 \leq t \leq T_0$.
As usual, we remove $xy \in E(G)$ if

1. $\{x, y\} \cap V_0 \neq \emptyset$,
2. $\{x, y\} \subseteq V_i$ for some $i \in [t]$,
3. $xy \in E(V_i, V_j)$ and (V_i, V_j) is not an ε-regular pair,
4. $xy \in E(V_i, V_j)$ and $d(V_i, V_j) < d_0$.

We get that there are at most εn^2 edges of Type 1, at most

$$t\binom{\frac{n}{t}}{2} \leq \frac{n^2}{2t} \leq \frac{n^2}{2t_0}$$

edges of Type 2, at most

$$\varepsilon t^2 \left(\frac{n}{t}\right)^2 = \varepsilon n^2$$

edges of Type 3, and at most

$$\binom{t}{2} d_0 \left(\frac{n}{t}\right)^2 \leq \frac{d_0}{2} \cdot n^2$$

edges of Type 4. Therefore, we have removed at most

$$\left(\frac{1}{2t_0} + 2\varepsilon + \frac{d_0}{2}\right) n^2 \leq \frac{3}{4} \cdot \gamma n^2$$

edges. Note that $\gamma n^2/4 \geq 1$. So, as there are still more than $t_{r-1}(n)$ edges left, Turán's Theorem implies that there exists a K_r in the remaining graph. By our choice of which edges got removed, this implies that w.l.o.g. (V_i, V_j) is ε-regular with density at least d_0 for all $1 \le i < j \le r \le t_0$. Furthermore, note that for all $i \in [r]$ we have

$$|V_i| \ge \frac{(1-\varepsilon)n}{t} \ge \frac{n}{2T_0} \ge M \cdot s.$$

So, $K_s^r \subseteq G$ by the Embedding Lemma. □

Remark. Technically, the problems asks you to deduce the theorem as outlined in Diestel, but I can't be bothered to prove the Blow-Up Lemma, if we already did the Embedding Lemma in class anyway.

Sheet 9

Exercise 1

For fixed $l \ge 3$, the off-diagonal graph Ramsey number $R(K_l, K_n)$ polynomial in n.

Proof. We will concretely show that for $n, l \ge 2$ we have

$$R(K_l, K_n) \le \binom{n+l-2}{l-1}.$$

Then, for fixed $l \ge 3$, this bound is in particular polynomial in n.
We will prove this bound inductively. For $n = 2 = l$, we get $R(K_2, K_2) = 2 \le \binom{2+2-2}{1}$. So, for the induction hypothesis, consider $n, l \ge 2$ with $n + l > 2$ and assume that for all $n', l' \ge 2$ with $n' + l' < n + l$ the bound is true.
Let $N = R(K_l, K_n) - 1$. By definition, there is a red/blue-coloring c of K_N such that there is no red K_l nor blue K_n. Let v be an arbitrary vertex in $K_N, X = \{u \in V(K_N) \mid c(uv) = \text{red}\}$ and $Y = \{u \in V(K_N) \mid c(uv) = \text{blue}\}$. By definition of c,

$$|X| \le R(K_{l-1}, K_n) - 1 \qquad\qquad |Y| \le R(K_l, K_{n-1}) - 1.$$

It follows that $R(K_l, K_n) \le R(K_{l-1}, K_n) + R(K_l, K_{n-1})$ since

$$R(K_l, K_n) - 1 = N = |X \cup Y \cup \{v\}| + 1 \le R(K_{l-1}, K_n) - 1 + R(K_l, K_{n-1}) - 1 + 1.$$

Applying the induction hypothesis, we therefore get

$$R(K_l, K_n) \le \binom{n+l-3}{l-2} + \binom{n+l-3}{l-1} = \binom{n+l-2}{l-1}$$

as desired. This concludes the proof. □

Alternative proof. We will show that $R(K_l, K_n) \le n^{l-1}$. To do that, we proceed by induction on l: For $l \in \{1, 2\}$, the claim is trivial, so consider $l \ge 3$ and assume the statement holds for values smaller than l.

Let $N = n^{l-1}$ and consider an arbitrary coloring of K_N. Pick any vertex v_1 in K_N. v_1 can have at most $n^{l-2} - 1$ red, incident edges, as otherwise the corresponding vertices would either contain a red K_{l-1}, forming with v_1 a red K_l, or a blue K_n by the induction hypothesis. So, set V_1 to be the vertices that are adjacent to v_1 via a blue edge. Observe that $|V_1| \geq N - n^{l-2}$. Inductively, given V_i, we let $v_{i+1} \in V_i$ be arbitrary and have again, by the same argument, that at most $n^{l-2} - 1$ of the incident edges are red. In particular, setting $V_{i+1} \subseteq V_i$ to be those vertices in V_i, that are adjacent to v_{i+1} via a blue edge, if $|V_i| \geq N - i \cdot n^{l-2}$, $|V_{i+1}| \geq |V_i| - n^{l-2} \geq N - (i+1) \cdot n^{l-2}$ follows. This shows, that we can do this procedure at least n times. Hence, if at no step a red K_l or blue K_n is found, $\{v_1, \ldots, v_n\}$ induce a blue K_n at the end. □

Exercise 2

A family $\mathcal{F}$ of sets is called a *weak* Δ*-system* if every two sets have intersections of the same size and it is a Δ*-system* if every two sets intersect in the same set.

(i) For all integers $m, k \geq 2$ there exists an M such that every family $\mathcal{F}$ of k-element sets such that $|\mathcal{F}| = M$ contains a weak Δ-system $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| \geq m$.

(ii) For all integers $m, k \geq 2$ there exists an M such that every weak Δ-system $\mathcal{F}$ of k-element sets with $|\mathcal{F}| = M$ contains a Δ-system $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| \geq m$.

Elementary proof.

(i) We define

$$M \coloneqq R(\underbrace{m, \ldots, m}_{k \text{ times}})$$

and let $\mathcal{F}$ be a family of k-element sets such that $|\mathcal{F}| = M$. Let $\mathcal{F}$ be the vertices of K_M and color the edges with $\{0, 1, \ldots, k-1\}$ such that the color of AB, where $A, B \in \mathcal{F}$, is $|A \cap B|$. By definition of M, there exists a monochromatic clique of size at least m. The corresponding vertices then form the desired a weak Δ-system $\mathcal{F}' \subseteq \mathcal{F}$ of size at least m.

(ii) Consider weak Δ-systems $\mathcal{F}$ of k-element sets that are k'-intersecting. We show that

$$M = (m-1) \cdot \binom{k}{k'}$$

suffices.[18] So, let $\mathcal{F}$ be an arbitrary weak Δ-system of k-element sets that is k'-intersecting. Let $X \in \mathcal{F}$ be arbitrary. By the pigeonhole principle there is a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size at least $|\mathcal{F}| / \binom{k}{k'} \geq m - 1$ that all intersect X in the same way. Let $X' \subseteq X$ be the subset such that $X' = X \cap Y$ for all $Y \in \mathcal{F}'$. In other words, X' is a k'-sized subset of Y for all $Y \in \mathcal{F}'$. Hence, as $Y \cap Z \supseteq X'$ and $\mathcal{F}$ is k'-intersecting, we already have $Y \cap Z = X'$ for all $Y, Z \in \mathcal{F}'$.
So, $\mathcal{F}' \cup \{X\}$ is a desired Δ-system of size m. □

[18] Again, set $M = m \cdot \binom{k}{\lfloor k/2 \rfloor}$ for a universal bound.

Brute-force proof. We will just repeat the proof of the Sunflower Lemma. So, let's show that if a k-uniform family $\mathcal{F}$ has size at least $|\mathcal{F}| > k! \cdot (m-1)^k$, then a Δ-system $\mathcal{F}' \subseteq \mathcal{F}$ with $|\mathcal{F}'| \geq m$ exists: We will proceed by induction on $k \in \mathbb{N}$.

- If $k = 1$, k-uniform families is just a family of singleton sets that are in particular pairwise disjoint. So, $|\mathcal{F}| > m - 1$ suffices. ✓
- Consider $k > 1$ and assume that the statement is true for $k' < k$.
  Let $\mathcal{F}$ be a k-uniform family of size $|\mathcal{F}| > k! \cdot (m-1)^k$ and let $\{B_1, \ldots, B_l\} \subseteq \mathcal{F}$ be a largest subfamily of $\mathcal{F}$ such that all the B_i's are pairwise disjoint.
  If $l \geq m$, then we are done, so assume $l \leq m - 1$ and let $B = \bigcup_{i=1}^{l} B_i$. Obviously, every B_i intersects B and due to maximality, the same is true for all $F \in \mathcal{F}$.
  So, by the pigeonhole principle, there is $x \in B$ such that at least
  $$\frac{|\mathcal{F}|}{|B|} \geq \frac{|\mathcal{F}|}{k(m-1)} > (k-1)! \cdot (m-1)^{k-1}$$
  sets of $\mathcal{F}$ contain x. Let $\mathcal{F}_x$ be the family of sets in $\mathcal{F}$ that contain x. This family is $(k-1)$-uniform, so, applying the induction hypothesis, we get that $\mathcal{F}_x$ contains a Δ-system $\mathcal{F}'_x$ of size at least m. The sets that correspond to the in $\mathcal{F}'_x$ contains sets in $\mathcal{F}$ then form a Δ-system $\mathcal{F}'$.

This concludes the proof. □

Exercise 3 (Hales-Jewett Theorem $\implies$ Bipartite Induced Ramsey theorem)

For every bipartite graph $B = (X \cup Y, E)$ and every $n \geq 1$ the combinatorial lines in E^n correspond to induced copies of B in the bipartite graph B' defined by

$$V(B') = X^n \,\dot\cup\, Y^n \qquad E(B') = \{\{(x_1, \ldots, x_n), (y_1, \ldots, y_n)\} \mid \forall\, i \in [n] \colon x_i y_i \in E\}\,.$$

Proof. W.l.o.g. we may assume that B has no isolated vertices. Let $\mathcal{L}$ be a combinatorial line of E^n. Canonically, we claim that the induced copy of B is

$$V(B'') = \bigcup_{e \in \mathcal{L}} e = \underbrace{\left(X^n \cap \bigcup_{e \in \mathcal{L}}\right)}_{=:X''} \dot\cup \underbrace{\left(Y^n \cap \bigcup_{e \in \mathcal{L}}\right)}_{=:Y''} \qquad\qquad E(B'') = \mathcal{L}.$$

To be concrete, let $M \,\dot\cup\, C = [n]$, where $M \neq \emptyset$ denotes the moving part and C the constant part, and let $\varphi \colon C \to E$ be a map such that $\mathcal{L} = \{g_e \colon e \in E\}$

$$g_e \colon [n] \to E, i \mapsto \begin{cases} e, & i \in M \\ f(i), & i \in C. \end{cases}$$

From that, we see that

- all vertices of X'' have the same i-th component for all $i \in C$,
- all vertices of Y'' have the same i-th component for all $i \in C$,

- for every $v \in V(B'')$ all the j-th's component are the same for $j \in M$.

In addition, as every vertex is incident to at least one edge, the map $\psi\colon V(B'') \to V(B), v \mapsto v(j)$ for some $j \in M$ is well defined, a bijection and partite in the sense that $\psi(X'') = X$ and $\psi(Y'') = Y$.
We also see that $uv \in \mathcal{L}$ if and only if $u(j)v(j) \in E$ for all $j \in M$. As all those components are the same for $j \in M$, this is equivalent to $\varphi(u)\varphi(j) = u(j)v(j) \in E$. $\square$

Exercise 4

For all $g, k \in \mathbb{N}, l \geq 2$ there exists a k-uniform hypergraph H with $\chi(H) > l, g(H) > g$.

Proof. We do an induction on g for fixed l and always deal with all $k \in \mathbb{N}$ simultaneously:

- For the induction base $g = 1$, note that $H = K^{(k)}_{l(k-1)+1}$, where the latter denotes the k-uniform clique on $l(k-1)+1$ vertices, has chromatic number greater than l since any l-vertex coloring of H has, by the pigeonhole principle, one color class of vertices of size at least k and therefore the corresponding edge would be monochromatic. Trivially, the girth of H is greater than g.
- Assume that for all $k \in \mathbb{N}$ there exists a k-uniform hypergraph H' with $\chi(H') > l$ and $g(H') > g$.
- For the construction of a k-uniform hypergraph H such that $\chi(H) > l$ and $g(H) > g+1$, we will define a sequence of graphs / “pictures” where the final graph is our H. So, start out with the “picture” P_0 being the vertex-disjoint k-uniform edges that project onto the k-uniform clique $K^{(k)}_{l(k-1)+1}$. Let $V^0_1, \ldots, V^0_{l(k-1)+1}$ be the partitioning of the vertices of P_0 such that V^0_i contain all vertices that correspond to v_i in $K^{(k)}_{l(k-1)+1}$, where we think of
$$V\left(K^{(k)}_{l(k-1)+1}\right) = \left\{v_1, \ldots, v_{l(k-1)+1}\right\}.$$
In particular, each vertex class forms an independent set.
Now, to define P_i from P_{i-1}, consider the vertex class V^{i-1}_i. By the induction hypothesis there exists a $|V^{i-1}_i|$-uniform hypergraph H'_i such that $\chi(H'_i) > l$ and $g(H'_i) > g$. We will then let $V(H'_i)$ play the role of V^i_i and for every $|V^{i-1}_i|$-uniform edge in H', we let the vertices in that edge play the role of V^{i-1}_i and “glue” onto them a to the other vertices otherwise disjoint copy of P_{i-1}. Naturally, we then think of vertices that are copies of vertices in V^{i-1}_j as vertices of V^i_j for all $j \in [l(k-1)+1]$ and see that the vertex classes still form independent sets.
Finally, we let $H = P_{l(k-1)+1}$.
- The fact that $g(H) > g+1$ is a straightforward inductive argument. Indeed, P_0 is the union of disjoint k-uniform edges and therefore has no cycles. Now, assume that P_{i-1} has girth greater than $g+1$. Then, looking at P_i, a cycle that is completely contained in one of the copies of P_{i-1} must have length greater than $g+1$. So, consider a cycle $C = v_1, e_1, v_2, e_2, \ldots, v_r, e_r$ that traverses multiple copies

of P_{i-1}. Since $g(H'_i) > g$, C contains at least $g+1$ vertices in the vertex class V_i^i. However, as we don't actually contain the edges of H'_i and V_i^i is an independent set in P_i, we must "detour" using a vertex outside of V_i^i whenever we want to move from one vertex in V_i^i to the next one according to C. This means that $|V(C)| \geq 2g+1 > g+1$, so $g(P_i) > g+1$.

- Next, we need to show that $\chi(H) > l$. For that, let c be an arbitrary coloring. We will show that H contains a copy of P_0 where every vertex class is monochromatic. Then, by the pigeonhole principle, there must be a color class containing at least k of the vertex classes of P_0. In particular, the k-uniform edge that goes through k of those vertex classes would be monochromatic.
  So, start out with $H = P_{l(k-1)+1}$. By our choice of $H'_{l(k-1)+1}$, there must be $|V_{l(k-1)+1}^{(l(k-1))}|$ vertices in $P_{l(k-1)+1}$ that were assigned the same color and have a copy of $P_{l(k-1)}$ glued onto them. In general, iterating through, we see that for all $j = 1, \dots, l(k-1)+1$ there exists a copy of $P_{l(k-1)+1-j}$ such that the vertices corresponding to vertex class $V_{l(k-1)+1-i}^{l(k-1)+1-j}$ are monochromatic for all $i = 1, \dots, j$.

This concludes the proof. $\square$

Bonus Question: The Set-Theoretic Sunflower Lemma

Let $\mathcal{F} = (S_i)_{i \in I}$ be a sequence of finite sets such that I is uncountable. Then there is an uncountable $J \subseteq I$ and set K for which $S_i \cap S_j = K$ for every $i, j \in J, i \neq j$.

Proof. W.l.o.g. we may assume that $\mathcal{F}$ as a family of sets is k-uniform for some $k \in \mathbb{N}_0$. Indeed, if not then there must be an uncountable $J \subseteq I$ such that $(S_j)_{j \in J}$ is k-uniform for some $k \in \mathbb{N}_0$. Otherwise, as a countable union of countably many sets,

$$I = \bigcup_{n \in \mathbb{N}} \{i \in I \colon\ |S_i| = n\}$$

would be countable. ↯
So, let $\mathcal{F}$ be k-uniform for some $k \in \mathbb{N}_0$. From here, it is a similar argument to the usual proof of the Sunflower Lemma: We will now show the claim by induction on k.
If $k \in \{0, 1\}$, then the claim is trivial. So, let $k \geq 2$ and assume that for all smaller uniformities the statement is true. Let J' be a inclusion-maximal subset of I such that $(S_j)_{j \in J'}$ forms a family of pairwise disjoint sets. If J' is uncountable, we are done, so assume the opposite. Let $S = \bigcup_{j \in J'} S_j$ and note that S is countable. By maximality, S_i intersects S for all $i \in I$. By the same pigeonhole argument as above, there must be $x \in S$ such that $I' \subseteq \{i \in I \colon x \in S_i\}$ is uncountable. Let $\mathcal{F}' = (S_i \setminus \{x\})_{i \in I'}$. By the induction hypothesis, we have that there is an uncountable subset $J \subseteq I'$ such that there exists a set K' for which $(S_i \setminus \{x\}) \cap (S_j \setminus \{x\}) = K'$ for every $i, j \in J, i \neq j$.
In particular, $S_i \cap S_j = K$ for every $i, j \in J, i \neq j$, where $K = K' \cup \{x\}$. $\square$

Sheet 10

Exercise 1

Trees are not well-quasi ordered under the subgraph relation.

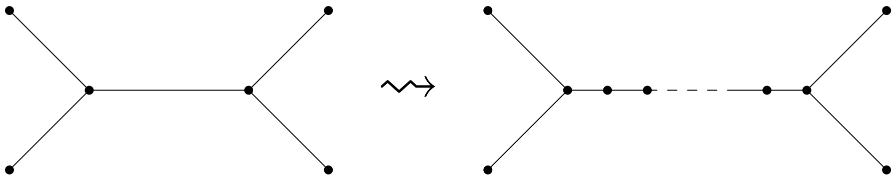

Figure 4: T_1 and general T_k

Proof. Consider the trees $(T_k)_{k\in\mathbb{N}}$ where T_1 is as given in Figure 4 and T_k results from subdividing the middle edge of T_1 $k-1$ times. Clearly, there are no $i<j$ such that $T_i \subseteq T_j$, as the vertices in T_i with degree 3 need to be matched with the vertices in T_j of the same degree, but the distance of those vertices in T_i is i while for T_j the distance is j. Hence, T_i can't be a subgraph of T_j for all $i<j$, so trees are not well-quasi ordered. □

Exercise 2

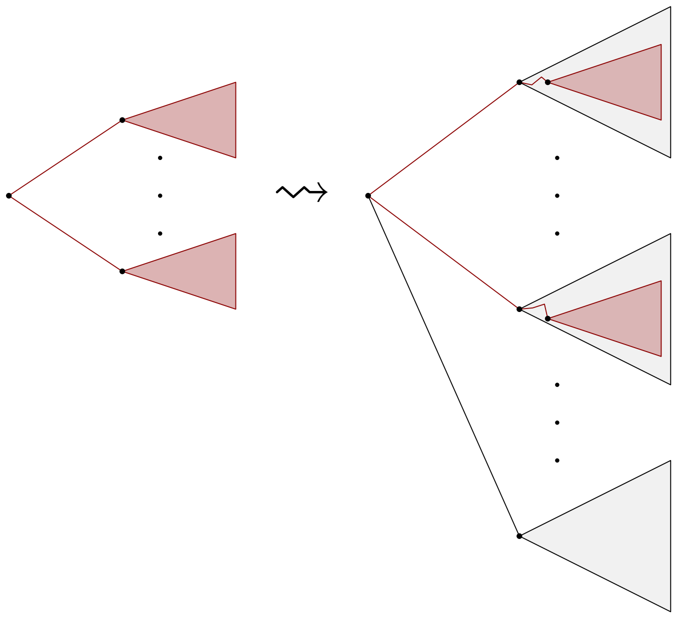

Figure 5: Proof sketch

Kruskal's Theorem on well-quasi orderings of rooted trees can be strengthened in such a way that roots are mapped to roots in the topological embedding.

Proof. Clearly, this relation defines a quasi-ordering, so we just need to check that every sequence is *good*[19]: Let $(T_k, r_k)_{k\in\mathbb{N}}$ be a sequence of rooted trees where r_k is the root of T_k. Furthermore, let A_k consist of the rooted subtrees corresponding to the children of the root r_k in T_k. Let $\leq$ be the order-preserving topological embedibility relation among finite rooted trees.[20]. By Kruskal's Theorem, we know that $\leq$ is a well-quasi ordering. As shown in Lemma 12.1.3, the by $\leq$ induced quasi-ordering on finite subsets of rooted trees is also a quasiorder. In particular, there are $i, j \in \mathbb{N}, i < j$, such that $A_i \leq A_j$.
Let $f\colon A_i \to A_j$ be the injective mapping such that $a \leq f(a)$ for all rooted subtrees $a \in A_i$. Using f, we can construct an order-preserving topological embedding that maps roots to roots as follows: Topologically embed each rooted subtree a in A_i into $f(a)$ while preserving the order and map r_i to r_j. As T_j is a tree, there is a unique path between r_j to the roots of each of the mapped to subtrees. Thus, this new mapping is again an order-preserving topological embedding that moreover maps r_i to r_j as desired. □

Exercise 3

The tree-width of a (non-empty) finite graph is at least its minimum degree.

Proof. Let G be a finite graph and $(T, \mathcal{V})$ be an optimal tree decomposition of G, i.e. the largest bag in $(T, \mathcal{V})$ has size $\mathrm{tw}(G) + 1$.

- If T has no leaves, i.e. consists of only one vertex, the corresponding bag must contain all vertices, of which there are at least $\delta(G) + 1$.
- If T has a *petal*, then there is a leaf $t \in V(T)$ such that $V_t \setminus V_s \neq \emptyset$ where s is the unique adjacent vertex of t, then let $x \in V_t \setminus V_s$. As $x \in V_s$, the third condition for tree decompositions forces V_t to be the only bag containing x. So, the second condition implies that all the neighbors of x are also in V_t.
  Thus, we get the desired bound, as
  $$\max_{t'\in T} |V_{t'}| = |V_t| \geq |N(x) \cup \{x\}| \geq \delta(G) + 1.$$
- If T has no *petals*, i.e. no leaves t such that $V_t \setminus V_s \neq \emptyset$, where s is the unique adjacent vertex of t, then let T' be the tree resulting from removing all leaves of T and let $\mathcal{V}' = \{V_t : t \in V(T')\}$. Then $(T', \mathcal{V}')$ is still an optimal tree decomposition of G. Indeed, every vertex and edge is by assumption still contained in one of the bags and restricting to a subtree of T also preserves the third condition (T3)[21] for tree decompositions.

If the third case occurs, we may do the same case distinction to $(T', \mathcal{V}')$ and as this last case can only happen finite many times, one of the first two cases must apply at some point, giving us the desired bound. □

[19] As defined in Chapter 12.1 of Diestel's *Graph Theory*.
[20] As defined in Chapter 12.2 of Diestel's *Graph Theory*.
[21] See Chapter 12.3 in Diestel's *Graph Theory*.

Exercise 4

Remark. A tree decomposition whose tree is a path is a *path decomposition.* The path-width of G is the least width of a path decomposition of G.

Trees have unbounded path width.

Proof. Concretely, we will show that $\mathrm{pw}(T_d) \geq d$ where T_d is the complete ternary tree of depth $d \in \mathbb{N}$. Then, the claim would follow. To be a little more precise, T_1 consists of a root with three children and T_k consists of a root with each of its three children being the root of a copy of T_{k-1}. We will proceed via induction on $d \in \mathbb{N}$:
For $d = 1$, T_d is a star and in particular contains edges, giving us $\mathrm{pw}(T_d) \geq 1$.
Now, consider $d > 1$ and let $(P, \mathcal{V})$ be an optimal path decomposition of T_d. Let F, G, H denote the three copies of T_{d-1} adjacent to the root of T_d. As P naturally induces path decompositions of F, G, H, the induction hypothesis implies that there are bags $V_F, V_G, V_H \in \mathcal{V}$ that contain at least d vertices of its respective subtree. W.l.og. we may assume that V_F, V_G, V_H are distinct, as otherwise we would be done.
Furthermore, let $V'_F, V'_G, V'_H \in \mathcal{V}$ denote the bags that contains the edge between the roots of T_d and the respective subtree. Again, we may assume that $\{V_F, V_G, V_H\} \cap \{V'_F, V'_G, V'_H\} = \emptyset$ as otherwise we would be done.

Case 1: One of the bags V_F, V_G, V_H, say V_F, lies in between two of V'_F, V'_G, V'_H in the path P. Then, by the third condition for tree decompositions, V_F must contain the root of T_d, so the largest bag of $(P, \mathcal{V})$ is at least $|V_F| \geq d + 1$.

Case 2: Otherwise, one of V_F, V_G, V_H, say V_F, must lie in between V_G and V'_G or V_H and V'_H, say V_G and V'_G. Indeed, at least two of V_F, V_G, V_H must be on the "same side" of the bags V'_F, V'_G, V'_H in P by the pigeonhole principle, so the "inner bag" must lie between the outer bag and its counterpart. However, by the third condition for tree decompositions, this implies that V_F contains the roof of G, so again the largest bag of $(P, \mathcal{V})$ is at least $|V_F| \geq d + 1$.

As we have concluded that $\mathrm{pw}(T_d) \geq d$ in both cases, we are done. □

Remark. This shows that the path-width grows at least logarithmically in the number of vertices for trees. I wondered whether there are more extreme subfamilies of trees that provide an even faster growth in the number of vertices. Asking Schacht the question I had in mind, i.e. what the asymptotic growth of

$$f(n) = \max \{\mathrm{pw}(T) \mid T \text{ tree with } |V(T)| = n\}$$

is, it was answered in the negative: $f(n) \in \Theta(\log(n))$. This was shown by Petra Scheffler in her paper *A Linear Algorithm for the Pathwidth of Trees.*