A Compact Summary of Model Theory

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1 First order logic

1.1 Structures

- A vocabulary τ contains constant symbols, relation symbols and function symbols. For every relation and function symbol we store their arity.
- A τ-structure $\mathcal{A}$ with underlying set $A \neq \emptyset$ gives each symbol a meaning.
- A τ-structure $\mathcal{B}$ is a substructure of $\mathcal{A}$ if $\mathcal{A}$ and $\mathcal{B}$ agree on $B \subseteq A$ and share the same interpretations for the constant symbols.
- In particular, for B to carry a substructure of $\mathcal{A}$, it must be closed under function applications and must contain the interpretation of the constant symbols.
- An isomorphism $i \colon \mathcal{A} \to \mathcal{B}$ is a bijection $A \to B$ such that $f^{\mathcal{B}} \circ i$" = "$i \circ f^{\mathcal{A}}$, $i(c^{\mathcal{A}}) = c^{\mathcal{B}}$ for all constant symbols c and $R^{\mathcal{B}}(i(a_1), \ldots, i(a_n)) \iff R^{\mathcal{A}}(a_1, \ldots, a_n)$.

1.2 Formulas

- Define terms inductively.
- Setting terms equal is a formula. ($\leftarrow$ atomic)
- Putting terms into a relation is a formula. ($\leftarrow$ atomic)
- If φ, ψ are formulas, then $\varphi \vee \psi$ and $\neg\varphi$ are formulas.
- If x is a variable and φ a formula, then $\exists x\varphi$ is a formula.
- x occurs freely in φ if it is not in scope of any quantifier.
- A formula without free variables is called a sentence.
- A set of (τ-)sentences is called a theory.

1.3 Semantics

Terms need to be evaluated in the obvious way and the rest is history. Don't forget though that notationally $(\mathcal{A}, a)$ being a model of Φ is equivalent to $\mathcal{A} \models \Phi[a]$. We don't require an assignment a if Φ is a theory. Also, if for every model $(\mathcal{A}, a)$ of Φ we have $\mathcal{A} \models \Psi[a]$, then $\Phi \models \Psi$.

1.4 Completeness

Theorem (Gödel's Completeness Theorem)**.** Let Φ be a theory over τ and φ a τ-sentence. Then

$$\Phi \models \varphi \iff \Phi \vdash \varphi.$$

In other words, everything that is true (in a model theoretic sense) can be proven and everything that we can prove is also true.

Theorem (Equivalent formulation)**.** A theory Φ is consistent (in a proof theoretic sense) if and only if Φ has a model.

1.5 Examples

A theory is a set of τ-sentences. To get the theory of a structure $\mathcal{A}$, we write $\mathrm{Th}(\mathcal{A})$ and set $\mathrm{Th}(\mathcal{C}) = \bigcap_{\mathcal{A}\in\mathcal{C}} \mathrm{Th}(\mathcal{A})$ if $\mathcal{C}$ is a class of structures. Given a theory Φ, we set $\mathrm{Mod}(\Phi)$ as the class of models $\mathcal{A}$ with $\mathcal{A} \models \Phi$.
The Peano Axioms, apart from rules for addition and multiplication, state that 0 is not the successor of any element, if two elements have the same successor, then they are the same element, and lastly that $\mathbb{N}$ satisfies induction: If A is closed under S and contains 0, then $A = \mathbb{N}$. In first order logic we only do this for definable A, i.e. where $A = \{x \colon \varphi(x, a_1, \ldots, a_n)\}$ holds for some formula φ and some $a_1, \ldots, a_n \in \mathbb{N}$. Note that here $0 \in \mathbb{N}$.

1.6 Compactness

Apart from completeness, we also like first order logic because we have compactness.

Theorem (Finiteness Theorem)**.** If for a theory Φ and a sentence φ we have $\Phi \vdash \varphi$, then there exists finite $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \vdash \varphi$. In other words, to prove one statement, a proof only requires only a finite portion of the axioms Φ.

Note that due to completeness the same is true with $\models$ instead of $\vdash$. The Finiteness Theorem can be shown using the Compactness Theorem.

Theorem (Compactness Theorem)**.** A theory Φ has a model if and only if every finite subset of Φ has a model.

We will prove this without any mentions of proof theory.

Definition (Filter)**.** Let I be a non-empty set. A filter $\emptyset \neq \mathcal{F} \subseteq \mathcal{P}(I)$ satisfies:
- $\emptyset \notin \mathcal{F}$. ("Most elements are not in the empty set.")
- $A \in \mathcal{F} \wedge B \in \mathcal{F} \implies A \cap B \in \mathcal{F}$. ("If most elements are in A and in B, then most elements are in $A \cap B$.")
- $A \in \mathcal{F} \wedge A \subseteq B \subseteq I \implies B \in \mathcal{F}$. ("If most elements are in A and $A \subseteq B$, then most elements are in B.")

We say $\mathcal{F}$ is an ultrafilter if $\mathcal{F}$ is maximal with respect to inclusion.

Every filter satisfies the finite intersection property, i.e. the intersection of finitely many elements in the filter is always non-empty. We can also go the other way around.

Lemma. Every family of subsets of I satisfying the finite intersection property can be extended to an ultrafilter.

Lemma (Characterization of ultrafilters)**.** A filter $\mathcal{F}$ of I is an ultrafilter if and only if one of the following conditions hold:
- For every $A \subseteq I$ we have $A \in \mathcal{F} \iff I \setminus A \in \mathcal{F}$.
- For every $A, B \subseteq I$, if $A \cup B \in \mathcal{F}$, then $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

In particular, an ultrafilter containing a finite set if always then of the form $\{S \subseteq I \colon i_0 \in S\}$ for some $i_0 \in I$. If this isn't the case, we call an ultrafilter free.
Notationwise, the choice I for the ground set is not coincidental, as it later is the index set: We are now in the situation that we have a family $(\mathcal{A}_i)_{i\in I}$ of τ-structures. For $a_1, \ldots, a_n \in \prod_{i\in I} A_i$ and a τ-formula $\varphi(x_1, \ldots, x_n)$ we let $[\![\varphi(a_1, \ldots, a_n)]\!] = \{i \in I \colon \mathcal{A}_i \models \varphi(a_1(i), \ldots, a_n(i))\}$. Let $\mathcal{U}$ now be an ultrafilter. We set $a \sim_{\mathcal{U}} b$ for $a, b \in \prod_{i\in I} A_i$ if $[\![a = b]\!] \in \mathcal{U}$.

Lemma ($\sim_{\mathcal{U}}$ is nice.)**.** Let $\mathcal{U}$ be an ultrafilter of I and $(\mathcal{A}_i)_{i\in I}$ a family of τ-structures.
- $\sim_{\mathcal{U}}$ is an equivalence relation.
- If $a_i \sim_{\mathcal{U}} b_i$, then $f(a_1, \ldots, a_n) \sim_{\mathcal{U}} f(b_1, \ldots, b_n)$. (Formally, $[\![f(a_1, \ldots, a_n) = f(b_1, \ldots, b_n)]\!]) \in \mathcal{U}$.)
- If $a_i \sim_{\mathcal{U}} b_i$, $[\![R(a_1, \ldots, a_n)]\!] \in \mathcal{U}$ if and only if $[\![R(b_1, \ldots, b_n)]\!] \in \mathcal{U}$.

Definition (Ultraproduct)**.** Given the family and the ultrafilter, the ultraproduct $\mathcal{A} = \prod_{i \in I} \mathcal{A}_i$ is defined by $A = \prod_{i\in I} A_i / \sim_{\mathcal{U}}, c^{\mathcal{A}} = [i \mapsto c^{\mathcal{A}_i}]_{\mathcal{U}}, \forall\, a_1, \ldots, a_n \in \prod_{i \in I} A_i$:

$$f^{\mathcal{A}}([a_1]_{\mathcal{U}}, \ldots, [a_n]_{\mathcal{U}}) = [i \mapsto f^{\mathcal{A}_i}(a_1(i), \ldots, a_n(i))]_{\mathcal{U}}, ([a_1]_{\mathcal{U}}, \ldots, [a_n]_{\mathcal{U}}) \in R^{\mathcal{A}} :\Longleftrightarrow [\![R(a_1, \ldots, a_n)]\!] \in \mathcal{U}.$$

Theorem (Łoś's Theorem)**.** $\mathcal{A} \models \varphi([a_1]_{\mathcal{U}}, \ldots, [a_n]_{\mathcal{U}}) \iff [\![\varphi(a_1, \ldots, a_n)]\!] \in \mathcal{U}$

Theorem (Applications)**.**

- Infinite Ramsey Theorem $\implies$ Finite Ramsey Theorem
- If Φ has arbitarily large finite models, then Φ also has an infinite model.
- If φ holds in fields of arbitrarily large characteristic, then φ holds in a field of characteristic zero. ($\iff$ If φ holds for every field of characteristic zero, then there is $p > 0$ such that φ holds in every field of characteristic $\geq p$.)

1.7 Elementary Substructures

Definition. Let $\mathcal{A}$ be a substructure of $\mathcal{B}$. We say that $\varphi(x_1, \ldots, x_n)$ is absolute if for every $a_1, \ldots, a_n \in \mathcal{A}$

$$\mathcal{B} \models \varphi(a_1, \ldots, a_n) \iff \mathcal{A} \models \varphi(a_1, \ldots, a_n).$$

We then write $\mathcal{A} \preccurlyeq \mathcal{B}$. Furthermore, if $\mathcal{A}$ and $\mathcal{B}$ are arbitrary τ-structures that satisfy the same sentences, then we say that they are elementarily equivalent and write $\mathcal{A} \equiv \mathcal{B}$. Note that

$$\mathcal{A} \preccurlyeq \mathcal{B} \implies \mathcal{A} \equiv \mathcal{B}.$$

Lemma (Tarski-Vaught-Criterion)**.** Let $\mathcal{A}$ be a substructure of $\mathcal{B}$. Then $\mathcal{A} \preccurlyeq \mathcal{B}$ if and only if for all formulas $\varphi(x, x_1, \ldots, x_n)$ and all $a_1, \ldots, a_n \in A$ the following holds: If there exists $a \in B$ such that $\mathcal{B} \models \varphi(a, a_1, \ldots, a_n)$, then there is $a \in \mathcal{A}$ such that $\mathcal{B} \models \varphi(a, a_1, \ldots, a_n)$.

In other words, the only obstruction to being an elementary substructure is the absoluteness for existence in the non-trivial direction.

Theorem (Downward Löwenheim-Skolem Theorem)**.** Let $\mathcal{A}$ be a τ-structure and κ be an infinite cardinal such that $\kappa \geq |\tau|$. If $X \subseteq A$ satisfies $|X| \leq \kappa$, then there exists a elementary substructure of $\mathcal{A}$ that extends X and is at most of size κ.

The idea are Skolem functions, i.e. for every formula $\varphi(x, x_1, \ldots, x_n)$ let $f_\varphi(a_1, \ldots, a_n)$ be a witness to $(\exists\, x\varphi)(a_1, \ldots, a_n)$ if such a witness exists, otherwise arbitrary.

Theorem (Upward Löwenheim-Skolem Theorem (Weak Version))**.** Let Φ be a theory with an infinite model. Then Φ has arbitrarily large infinite models. In particular, for every infinite structure $\mathcal{A}$ there exist arbitrarily large models that are elementary equivalent to $\mathcal{A}$.

Definition. We say $\mathcal{A}$ is embeddable into $\mathcal{B}$, if $\mathcal{A}$ is isomorphic to an substructure of $\mathcal{B}$. We call the embedding elementary if the substructure of $\mathcal{B}$ to which we map to is even an elementary substructure of $\mathcal{B}$. If $\tau \subseteq \sigma$ are vocabularies and $\mathcal{A}, \mathcal{A}'$ are τ- / σ-structures that "agree" on τ, then $\mathcal{A}$ is the reduct of $\mathcal{A}'$ and $\mathcal{A}'$ is the expansion of $\mathcal{A}$.
Let $\mathcal{A}$ be a fixed τ-structure and consider $\tau \,\dot\cup\, \{c_a \mid a \in A\}$.

$$\begin{aligned}\operatorname{diag}(\mathcal{A}) &= \{\varphi(c_{a_1}, \ldots, c_{a_n}) \mid \varphi \text{ atomic or } \neg\varphi \text{ atomic}, a_1, \ldots, a_n \in A, \mathcal{A} \models \varphi(a_1, \ldots, a_n)\} \\ \operatorname{eldiag}(\mathcal{A}) &= \{\varphi(c_{a_1}, \ldots, c_{a_n}) \mid \varphi \text{ any formula}, a_1, \ldots, a_n \in A, \mathcal{A} \models \varphi(a_1, \ldots, a_n)\}\end{aligned}$$

The definitions are chosen in such a way that $\mathcal{A}$ always embeds into a model $\mathcal{B}$ of $\operatorname{diag}(\mathcal{A})$ and always elementary embeds into a model $\mathcal{B}$ of $\operatorname{eldiag}(\mathcal{A})$ via $a \mapsto c_a^{\mathcal{B}}$. (You can check the absoluteness for every formula via a chain of equivalences.) Note that those theories are consistent since we can expand $\mathcal{A}$ to a model of both of them. Note that the latter is also complete.

Theorem (Upward Löwenheim-Skolem Theorem (Strong Version))**.** Let $\mathcal{A}$ be an infinite τ-structure. Then there are arbitrarily large models in which $\mathcal{A}$ is elementary embeddable.

Theorem (Löwenheim-Skolem Theorem)**.** Let κ be an infinite cardinal with $\kappa \geq |\tau|$ and let Φ be a theory that has an infinite model. Then Φ has a model of size κ.

Implications:

- The theory of natural numbers $\mathcal{N} = (\mathbb{N}, 0, S, +.\cdot)$ (by which we really mean $\mathrm{Th}(\mathcal{N})$ and not just the deductive closure of the first order Peano Axioms) has arbitrarily large infinite models in which the standard natural numbers can be embedded. Even when only considering countably infinite models, there are some which are non-standard: Simply take a bigger non-standard model and apply the Downward Löwenheim-Skolem on a countable set X that contains both $\mathbb{N}$ and some non-standard element.
- ZFC is a first order theory with one relation symbol: "$\in$". If ZFC is a consistent theory, then by the Completeness Theorem and Löwenheim-Skolem it follows that there is a countable model of set theory. In this model there are only countably many elements that the model considers to be real numbers, even though the model itself believes that there are uncountably many real numbers. But this just means that the notion "countable" of the model is different from the corresponding notion in the real world, for instance because the model does not know the bijection between the countably many real numbers of the model and the natural numbers of the model that exists in the real world.

Definition.

- A poset $(I, \leq)$ is directed if there is always a join, i.e. $\forall\, i, j \in I \exists\, k \in I \colon i \leq k \wedge j \leq k$.
- An elementary directed system $(\mathcal{A}_i)_{i \in I}$ is a family of τ-structures with a directed index set such that for all $i, j \in I, i \leq j$, $e_{i,j} \colon \mathcal{A}_i \to \mathcal{A}_j$ is an elementary embedding and the embeddings "agree", i.e. if $i \leq j \leq k$ then $e_{i,k} = e_{j,k} \circ e_{i,j}$.
- We say that $\mathcal{A}$ is the limit to an elementary directed system $(\mathcal{A}_i)_{i \in I}$ if there are elementary embeddings $e_i \colon \mathcal{A}_i \to \mathcal{A}$ for all $i \in I$ that "agree" with the other embeddings, i.e. $e_i = e_j \circ e_{i,j}$ for all $i, j \in I, i \leq j$.

Theorem. Every elementary directed system has a limit.

The idea is to say that w.l.o.g. the A_i's are pairwise disjoint and then define for $i, j \leq k$ that $A_i \ni a \sim b \in A_j$ if $e_{i,k}(a) = e_{j,k}(b)$. This defines an equivalence relation. The limit $\mathcal{A}$ then has the underlying set $A = \biguplus_{i \in I} A_i / \sim$. $c^{\mathcal{A}}$ is defined as the equivalence class of $c^{\mathcal{A}_i}$ for some $i \in I$. For $a_1 \in A_{i_1}, \ldots, a_n \in A_{i_n}$ and $i \leq j$, let

$$f^{\mathcal{A}}([a_1], \ldots, [a_n]) = \left[f^{\mathcal{A}_j}(e_{i_1,j}(a_1), \ldots, e_{i_n,j}(a_n))\right]$$

and $R^{\mathcal{A}}([a_1], \ldots, [a_n])$ if and only if $R^{\mathcal{A}_j}(e_{i_1,j}(a_1), \ldots, e_{i_n,j}(a_n))$. That the e_i's ($a \mapsto [a]$) defines an embedding is obvious and for an elementary embedding, it suffices to consider as in Tarki-Vaught an absolute formula $\varphi(x, x_1, \ldots, x_n)$ and show that $(\exists\, x \varphi)$ is also absolute.

Corollary (Tarski's Chain Theorem)**.** If $(I, \leq)$ is a linear order and $(\mathcal{A}_i)_{i \in I}$ a family of τ-structures such that $\mathcal{A}_i \preccurlyeq \mathcal{A}_j$ for all $i, j \in I, i \leq j$, then there exists a τ-structure $\mathcal{A}$ on $A = \bigcup_{i \in I} A_i$ such that $\mathcal{A}_i \preccurlyeq \mathcal{A}$ for all $i \in I$.

2 Properties of first order theories

2.1 Categoricity and completeness

Definition. Let Φ be a theory over τ.

- $\mathrm{Ded}(\Phi) = \{\varphi \colon \varphi\ \tau\text{-sentence}, \Phi \models \varphi\}$ is the deductive closure.
- Φ is deductively closed, if $\Phi = \mathrm{Ded}(\Phi)$.

- Φ is complete if Φ is consistent and for every τ-sentence φ we have either $\varphi \in \Phi$ or $\neg\varphi \in \Phi$.
- Let κ be a cardinal. We say that Φ is κ-categorical if there is (up to isomorphism) a unique Φ-model of size κ.

Consequences:

- A complete theory is always deductively closed.
- The theory of a structure is always complete.

Lemma (Categoricity " $\implies$ " Completeness)**.** Let Φ be a **deductively closed** theory that has **only infinite models**. If Φ κ-categorical for some $\kappa \geq |\tau|$, then Φ is complete.

Proof by contradiction.

Theorem (Morley)**.** If Φ is a countable (i.e. $|\tau| = \aleph_0$) first order theory that is κ-categorical for some uncountable κ, then Φ is κ-categorical for all uncountable κ.

Theory of ...	Complete?	$\aleph_0$-categorical?	κ-categorical for uncountable κ?
Infinite sets	✓	✓	✓
$\mathbb{Q}$-vector spaces	✓		✓
Linear dense orders without endpoints	✓	✓	Nope, $(\mathbb{R}, \leq)$ and $(\mathbb{R} \setminus \{0\}, \leq)$.
$A = \mathbb{N} \times \{0\} \cup \mathbb{N} \times \{1\}$, $(a, b) \leq (x, y)$ if $b = y = 0$ and $a \leq x$	✓		
Random graphs	✓	✓	
$\textsc{AlgebraicallyClosedFields}_p$	✓		✓

2.2 Quantifier elimination

Definition. Let Φ be a theory. Formulas φ and ψ are Φ-equivalent if $\Phi \models \varphi \leftrightarrow \psi$. Φ has quantifier elimination if every formula φ is Φ-equivalent to a quantifier-free formula. A structure has quantifier elimination if its theory has.

Lemma. Φ has quantifier elimination if and only if for every quantifier free formula $\varphi(x, y_1, \ldots, y_n)$ the formula $(\exists\, x\varphi)(y_1, \ldots, y_n)$ is Φ-equivalent to a quantfier-free formula.

Lemma. Let Φ be a **complete** theory over a **finite relational** (i.e. without function symbols) vocabulary τ. Φ has quantifier elimination if and only if for every model $\mathcal{A}$ the following holds: If $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ satisfy the same atomic formulas, then for every $a_{n+1} \in A$ there exists $b_{n+1} \in A$ such that $(a_1, \ldots, a_{n+1})$ and $(b_1, \ldots, b_{n+1})$ satisfy the same atomic formulas.

The idea is that due to restrictions of the vocabulary, there are only finitely many formulas on variables $x_1, \ldots, x_n$. So, you can encode for every $(a_1, \ldots, a_n) \in A^n$ in one quantifier-free formula "This tuple satisfies the same atomic formulas as $(a_1, \ldots, a_n)$.".
This lemma can be relaxed by either removing the condition that Φ is complete or by only needing the last statement to hold for (at least) one model.

Corollary.

- The theory of dense linear orders without endpoints has quantifier elimination.
- The theory of random graphs has quantifier elimination.

Theorem. Let Φ be a complete theory over a finite relational vocabulary τ that has infinite models and quantifier elimination. Then Φ is $\aleph_0$-categorical.

Lemma. Let Φ be a **complete** theory over **any** vocabulary. Then Φ has quantifier elimination if and only if the following holds: For any embeddings $f, g\colon \mathcal{C} \to \mathcal{M}$, where $\mathcal{C}$ is a τ-structure and $\mathcal{M}$ is a model of Φ, we have for all quantifier-free $\varphi(x, y_1, \ldots, y_n)$ and all $c \in C^n$ that

$$\mathcal{M} \models (\exists\, x\varphi)(f(c_1), \ldots, f(c_n)) \iff \mathcal{M} \models (\exists\, x\varphi)(g(c_1), \ldots, g(c_n)).$$

For the backward implication, use contraposition.

Theorem. The theory of algebraically closed field of fixed characteristic p has quantifier elimination.

Definition. A theory Φ is model-complete if for every model $\mathcal{B}$ and substructure $\mathcal{A}$ that is also a model of Φ we have that $\mathcal{A} \preccurlyeq \mathcal{B}$.

Lemma. If a theory Φ has quantifier elimination ~~and is complete~~[1], then Φ is model-complete.

Corollary. The theory of algebraically closed field of fixed characteristic p is model-complete.

Theorem (Hilbert's Basissatz)**.** If R is a Noetherian ring, then $R[X]$ is also a Noetherian ring.

Noetherian here means that every submodule of R is finitely generated.[2]

Theorem (Hilbert's Nullstellensatz)**.** Let F be an algebraically closed field and $I \subset F[X_1, \ldots, X_n]$ be a proper ideal. Then there is $x \in F^n$ such that for all $p(X) \in I$ we have $p(x) = 0$.

Lemma. Let Φ be **any** theory over **any** vocabulary. Then Φ has quantifier elimination if and only if the following holds: For any embeddings $f \colon \mathcal{C} \to \mathcal{M}, g \colon \mathcal{C} \to \mathcal{N}$, where $\mathcal{C}$ is a τ-structure and $\mathcal{M}, \mathcal{N}$ are models of Φ, we have for all quantifier-free $\varphi(x, y_1, \ldots, y_n)$ and all $c \in C^n$ that

$$\mathcal{M} \models (\exists\, x\varphi)(f(c_1), \ldots, f(c_n)) \iff \mathcal{N} \models (\exists\, x\varphi)(g(c_1), \ldots, g(c_n)).$$

The proof of the backward implication uses the first lemma: Let $\varphi(x, y_1, \ldots, y_n)$ be quantifier-free. We extend the vocabulary by contant symbols $c_1, \ldots, c_n$ to σ such that $(\exists\, x\varphi)(c_1, \ldots, c_n)$ is a sentence. Note then that if two σ-models of Φ satisfy the same quantifier-free sentences, then $(\exists\, x\varphi)(c_1, \ldots, c_n)$ holds in one model if and only if it holds in the other one.

Definition.
- An ordered ring $(R, 0, 1, +, \cdot, \leq)$ is a ring with an additional linear order on it that acts the way we're used to for real numbers.
- An ordered field is an ordered ring that is also a field.
- A real closed field is an ordered field, where every non-negative element has a squareroot and every odd polynomial a zero (in the field).

Lemma. If the theory of real closed fields has quantifier elimination, then it is also complete.

Lemma.
- Every ordered ring can via the field of fractions be extended into an ordered field, where the orders agree with each other.
- A field is formally real, if -1 can't be written as a sum of squares.
- Every formally real field has an order such that $a < 0$ if and only if a can't be written as a sum of squares.
- The order of a real closed field is uniquely determined by $a \leq b \iff \exists\, c \colon b = a + c^2$.
- Every ordered field has a real closure, i.e. it is a substructure of some real closed field, where every element of that field is algebraic over the ordered field.[3] This real closure is unique up to isomorphism over the base field.
- Adjoining a zero of $X^2 + 1$ to a real closed field yields an algebraically closed field.
- Over a real closed field, every polynomial in one variable decomposes into linear factors and quadratic factors of the form $(X + d)^2 + e$ where e is positive.

Lemma. The theory of real closed fields has quantifier elimination and is in particular model-complete.

Theorem (Hilbert's 17th Problem)**.** Let F be a real closed field. Then $f \in F[X_1, \ldots, X_n]$ can be written as a sum of squares of rational functions if and only if $f(a_1, \ldots, a_n) \geq 0$ for all $a_1, \ldots, a_n \in F$.

[1]Is said to be necessary in the lecture notes, but Wikipedia disagrees and it is nowhere used in the proof.
[2]Furthermore, we only consider the commutative case here.
[3]In particular, the real closure of $\mathbb{Q}$ is not $\mathbb{R}$. Great terminology!