Master Preparation Course

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1 Linear Algebra

Definition (Group)**.** A group G is a set, together with a map $\circ\colon G \times G \to G$ and identity element e such that

1. $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in G$ (associativity)
2. $e \circ a = a \circ e = a$ for all $a \in G$ (identity element)
3. for all $a \in G$, there is an $a' \in G$ such that $a \circ a' = a' \circ a = e$ (inverse element).

A group is called *abelian* (or *commutative*) if $a \circ b = b \circ a$ for all $a, b \in G$.

Notation. Groups are often written "multiplicatively" (with $a \circ b = a \cdot b, a' = a^{-1}, e = 1$), but abelian groups are sometimes written "additively" ($a \circ b = a + b, a' = -a, e = 0$).

Remark. The identity element e and the inverse of a given element $a \in G$ are unique.

Definition. A *group homomorphism* $\varphi\colon G \to H$ between groups G and H is a map (of sets) such that

$$\forall\, a, b \in G\colon \varphi(a) \circ \varphi(b) = \varphi(a \circ b).$$

Remark. Then also $\varphi(e_G) = e_H$ and $\varphi(a^{-1}) = \varphi(a)^{-1}$.

Example. The integers $\{0, \ldots, n-1\} \subseteq \mathbb{Z}$ become the *cyclic group* $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ with operation the addition modulo n. This group can be obtained as the quotient of the group $\mathbb{Z}$ (free group in one generator) of integers with addition by the subgroup in $n\mathbb{Z}$ of integers divisble by n.

Definition (Ring)**.** A *ring* R is a set with $\cdot, +\colon R \times R \to R$ and elements 0 (zero) and 1 (unit) such that

1. $(R, +, 0)$ is an abelian group,
2. $(R, \cdot, 1)$ is a monoid, i.e. $\cdot$ is associative with identity element 1 (but in general there are no multiplicative inverses),
3. $a \cdot (b + c) = a \cdot b + a \cdot c$ (distributivity).

A ring R is called *commutative* if $\cdot$ is commutative, i.e. if $a \cdot b = b \cdot a$ for all $a, b \in R$.

Remark. The subset of elements in R that possess a mulitplicative inverse form a group, the multiplicative group $R^\times$ of R.
$\to$ Recall the theory of ring homomorphisms, subrings, ideals (= kernels of ring homomorphisms).

Example. The integers $\mathbb{Z}$ with the $+, 0, \cdot, 1$ form a commutative ring. It is not a field, because 2 (for example) does not have a multiplicative inverse in $\mathbb{Z}$. The multiplicative group is $\mathbb{Z}^\times = \{\pm 1\} \simeq \mathbb{Z}_2$.

Definition (Field)**.** A *field* k is a set with $\cdot, +\colon k \times k \to k$ and elements $0, 1$ such that
- $(k, +, 0)$ is an abelian group,
- $(k \setminus \{0\}, \cdot, 1)$ is an abelian group (denoted by $k^\times$),
- $a \cdot (b + c) = a \cdot b + a \cdot c$ (distributivity).

(Equivalently, a field k is a ring, which
- is commutative,
- has a multiplicative inverse a^{-1} for all $a \in k \setminus \{0\}$.)

Remark. It is easy to see that the identity element 0 of addition can never have a multiplicative inverse 0^{-1}, so in a sense, $k \setminus \{0\}$ is a maximal choice.
$\to$ Recall that fields do not have non-trivial ideals and quotients, so every field homomorphism is injective. Recall the theory of subfields and field automorphisms.

Teaser. Linear Algebra is just the representation theory of fields.

Example. $\mathbb{Q} = \mathrm{Quot}(\mathbb{Z})$ ("quotient construction"), $\mathbb{R}$ (e.g. as equivalence classes of Cauchy sequences in $\mathbb{Q}$), $\mathbb{C} = \mathbb{R} \oplus \mathrm{i}\,\mathbb{R}$.

Definition (Vector Spaces)**.** A *vector space* V over a field k is an abelian group $(V, +, 0)$ with scalar multiplication $k \times V \to V$ such that
- $1_k v = v, 0_k v = 0_V$ for all $v \in V$,
- $(ab)v = a(bv), (a + b)v = av + bv$ for all $a, b \in k, v \in V$,
- $a(v + w) = av + aw$ for all $a \in k, v, w \in V$.

(Note that if k is just a ring the same axioms define a k-module.)

Example. For every field k, the space $k^n (= k \oplus \cdots \oplus k)$ of n-tuples

$$v = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

with entry-wise addition and entry-wise scalar multiplication defines a vector space over k. There is a standard basis $\{e_i\}_{i=1,\dots,n}$ with $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ (with the 1 at the i-th entry) and hence k^n has dimension n over k.

Definition (Basis)**.** A basis of a vector space V over a field k is a set of elements B such that

- no proper k-linear combinations of basis elements is 0, (linear independence)
- every element in V can be written as a k-linear combination of basis elements (spanning property).

Proposition. Every vector space has a (possibly infinite) basis.

(Note that the axiom of choice is needed for "large" infinite bases. Further note that this theorem is in general false for a module V over a ring k.)

Proposition. Every basis of a vector space V has the same cardinality, which we call the *dimension* $\dim(V)$ of V.

Example. Consider the space of continuous functions $\mathcal{C}^0(M)$ from a topological space (or more narrowly manifold or submanifold) with values in $\mathbb{R}$ (or $\mathbb{C}$). We find two structures

- It is a commutative ring with point-wise addition and multiplication. (But like this, only functions that are non-zero everywhere are invertible.)
- There is a scalar multiplication by $\mathbb{R}$ (or $\mathbb{C}$) that forms $\mathcal{C}^0(M)$ (with point-wise addition) into a vector space over $\mathbb{R}$ (or $\mathbb{C}$).

(Note that a compatible combination of a ring and a vector space (like aove) over some base field k is called a k-algebra.)

Even though there exists a basis $\mathcal{C}^0(M)$, it is very hard to write one down. The first guesses of bases one might think of usually require infinite series, so they are no proper vector bases.

Definition (Direct Sum)**.** For vector spaces V, W over the same field k we define a vector space $V \oplus W$ called *direct sum* as the tuples (v, w), $v \in V, w \in W$ with component-wise addition and (scalar) multiplication. Note that if $B_V \subseteq V, B_W \subseteq W$ are bases, the union $B_V \cup B_W$ is a basis of $V \oplus W$ (embedded via $x \mapsto (x, 0)$ and $x \mapsto (0, x)$, respectively), but a basis of $V \oplus W$ does not have to be of this form.

Examples (From Quantum Mechanics)**.** In quantum mechanics, the space of states $\mathcal{H}$ is a vector space over the field $\mathbb{C}$. A state $\psi \in \mathcal{H}$ can thus be an arbitrary complex linear combination (of some base states). The absolute value squared of each coefficient stands fo an (unnormalized) probability and the complex angle of each coefficient is the so-called phase, which cannot be measured directly, but shows its effect when states are added (interference experiments).

- Polarized light: A light beam in Z-direction can be polarized in X- or Y-direction, leaving (apart from frequency) a 2-dimensional state space $\mathcal{H}$ with basis X, Y. A second, physically meaningfull basis is $X + \mathrm{i}\, Y, X - \mathrm{i}\, Y$, corresponding to right and left circularly polarized light.

- Ammonia: An ammonia molecule NH_3 looks like a tetrahedron. In a simple model, there are two states "up" and "down", but in nature one usually encounters a "50-50" superposition of both states with less total energy ($\rightsquigarrow$ see eigenvalues). The same is true for the two states of a benzene molecule C_6H_6.
- *Spatial position* (infinite-dimensional example): For a simple (spin-less) particle (and disregarding momentum) the state spacecan be take to be the infinite-dimensional $\mathcal{H} = \mathcal{C}^0(M)$, the continuous complex function on some topological space M. It expresses with which probability (and phase) the particle is at a certain point. There are different (non-proper) bases, such as the expansion of a function in "free waves" ($\leftarrow \mathrm{e}^{2\pi\,\mathrm{i}\,x\cdot v} = f(x)$).

Definition (Vector Space Homomorphism)**.** Let V and W be vector spaces over the same field k. A *homomorphism* $\varphi\colon V \to W$ of vector spaces is a map preserving the vector space structure, i.e. a *linear map*:

- $\varphi(av) = a\varphi(v)$ for all $a \in k, v \in V$,
- $\varphi(v + w) = \varphi(v) + \varphi(w)$ for all $v, w \in V$.

In other words, linear combinations are preserved.

Recall: As for groups, rings, fields, etc. we call φ a

- monomorphism if it is injective (denoted by $\hookrightarrow$),
- epimorphism if it is surjective (denoted by $\twoheadrightarrow$),
- isomorphism if it is bijective (denoted by $\cong$).

For $V = W$ we use the expressions

- endomorphisms for homomorphisms $V \to V$,
- automorphisms for isomorphisms $V \to V$.

Example. Every vector space V over a field k of finite dimension is isomorphic to k^n. This is easily proved by choosing a basis of V and constructing a (linear) map for all k-coefficients in this basis, which lie in k^n.

Note that the statement is still true for infinite dimensions with n as the cardinality of the basis.
However, these vector spaces are usually treated in a different way ($\to$ e.g. topological vector spaces). If we include this structure, there are then non-isomorphic infinite-dimensional vector spaces of the same cardinality.

Definition (Image and kernel)**.** Let V and W be vector spaces over k and $\varphi\colon V \to W$ a homomorphism of vector spaces. Then we define the following subvector spaces of V and W as follows:

- the *kernel* $\ker(\varphi) = \{v \in V \mid \varphi(v) = 0\} \subseteq V$,
- the *image* $\operatorname{im}(\varphi) = \{\varphi(v) \mid v \in V\} \subseteq W$.

Definition (Quotient Vector Space)**.** For any subvector space $U \subseteq V$ we define the *quotient vector space* V/U as the set of equivalence classes $\{[v] = v + U \mid v \in V\}$ with the equivalence relation $v \sim w$ iff $v - w \in U$, together with the induced addition and scalar multiplication.

Proposition. For every homomorphism φ of vector spaces the image $\operatorname{im}(\varphi)$ is isomorphic to $V/\ker(\varphi)$ as vector spaces.

Hence, in some sense, all morphisms φ look like a surjective map $V \twoheadrightarrow V/\ker(\varphi)$ composed with the injective map $V/\ker(\varphi) \cong \operatorname{im}(\varphi) \hookrightarrow W$.
As a consequence, $\dim(V) = \dim(\ker(\varphi)) + \dim(\operatorname{im}(\varphi))$.

Example. The following argument is useful for other categories as well (groups, rings, ...): Suppose there is a surjective vector space homomorphism $\varphi\colon U \twoheadrightarrow V$ and another homomorphism $\psi\colon U \to W$.
Question: When does there exists a factorisation $\chi\colon V \to W$, i.e. $\psi = \chi \circ \varphi$?

$$\begin{tikzcd}
U \arrow[rr, "\psi"] \arrow[dr, "\varphi"'] & & W \\
& V \arrow[ur, dashed, "\chi"'] \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

We would like to define $\chi(v)$ as follows: Take some choice $u \in U$ with $\varphi(u) = v$ (possible since φ is assumed to be surjective) and define $\chi(v) := \psi(u)$.
For χ to be well-defined, the expression $\psi(u)$ must be independent of the choice of $u \in U$ with $\varphi(u) = v$ which may be shifted by an element in $\ker(\varphi) \subseteq U$. Hence, χ is well-defined iff $\ker(\varphi) \subseteq \ker(\psi)$.

Proposition (Factorisation)**.** Let $\varphi\colon U \to V$ be a surjective vector space homomorphism and $\psi\colon U \to W$ another homomorphism. A factorisation $\chi\colon V \to W$ (meaning $\chi \circ \varphi = \psi$) exists iff $\ker(\varphi) \subseteq \ker(\psi)$. This vector space homomorphism χ is surjective iff ψ is, and χ is injective iff $\ker(\varphi) = \ker(\psi)$.

Proposition (Base Change)**.** Every basis $B = \{v_1, \ldots, v_n\}$ of a finite-dimensional vector space V over a field k induces a vector space isomorphism φ_B to k^n sending every vector $v = \sum_{i=1}^n a_i v_i$ to its coefficients $(a_1, \ldots, a_n)^\top$ in the basis B.

Suppose $B' = \{v'_1, \ldots, v'_n\}$ is another basis of V, then we would like to calculate the homomorphism of the coefficients of a vector v from the old basis to the new one B':

$$k^n \xleftarrow{\varphi_B} V \xrightarrow{\varphi_{B'}} k^n, T_{B \to B'} = \varphi_{B'} \circ \varphi_B^{-1}\colon k^n \to k^n.$$

Normally, $T_{B \to B'}$ is an $(n \times n)$-matrix with columns as follows: The j-th column is the j-th old basis vector v_j expressed in the new basis. (And analogously for $T_{B \to B'}^{-1} = T_{B' \to B}$).
Recall that for a vector space homomorphism $V \to W$ the corresponding matrix $(A_{i,j})_{i,j}$ transforms according to a base change in V, W or both as follows:

$$A \mapsto T_{B_W \to B'_W} \cdot A \cdot T_{B_V \to B'_V}^{-1}.$$

Example. Consider the vector space $V = k[X]$ over a field k. The differential operator ∂_x is a vector space endomorphism of V. The kernel is the 1-dimensional subspace of constants $\ker(k) = k \subseteq k[X]$, while the image is the full vector space $\operatorname{im}(\partial_x) = k[X]$. If we choose a basis of monomial $v_i = X^i$, the (infinite) representing matrix of ∂_x is:

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & 0 & 2 & 0 & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & 0 & 3 & 0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & 0 & 4 & 0 & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & 0 & 5 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Definition (Determinant)**.** Let $\varphi\colon V \to V$ be an endomorphism of the n-dimensional vector space V over k. The determinant $\det\varphi$ of φ is an element of k defined for a representing matrix

$$\det\varphi = \sum_{\sigma\in S_n} \operatorname{sgn}\sigma \prod_{I=1}^{n} A_{\sigma(i),i}.$$

Recall the following facts:

- $\det\varphi\circ\psi = \det\varphi\det\psi$,
- $\det\varphi$ is independent of the choice of the base B of V and hence of the representing matrix A of φ because

$$\begin{aligned} \det(A') &= \det(T_{B_W\to B'_W}\cdot A\cdot T^{-1}_{B_W\to B'_W}) \\ &= \det(T_{B_W\to B'_W})\cdot\det(A)\cdot\det(T^{-1}_{B_W\to B'_W}) \\ &= \det(A). \end{aligned}$$

- $\det\varphi$ measures the "volume ratio" of φ. In particular, $\det 1_V = 1$ and $\det\varphi \neq 0$ iff φ is an automorphism of V.

As lower dimensional examples, we have:

- $n = 1$: $\det A = A_{1,1}$,
- $n = 2$: $\det A = A_{1,1}A_{2,2} - A_{1,2}A_{2,1}$,
- $n = 3$: Sarrus' formula.

Definition (Tensor Product (via "universal construction"))**.** Let V, W be k-vector spaces over a field k. A *tensor product* of V and W is a vector space $V\otimes_k W$ together with a bilinear map $\otimes\colon V\times W \to V\otimes_k W, (v,w)\mapsto v\otimes w$ such that for any bilinear map $f\colon V\times W\to Z$ there is a *unique* factorization $\tilde{f}\colon V\otimes_k W\to Z$ (i.e. $\tilde{f}\circ\otimes = f$).

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[dr, "f"'] & V \otimes_k W \arrow[d, dashed, "\tilde{f}"] \\ & Z \end{tikzcd}$$

Remark. The tensor product is *unique* if it exists, which follows by a straightforward argument for universal constructions: Suppose there is a second tensor product $(V \otimes_k' W, \otimes')$, then we can apply the universal property

- of $V \otimes_k W$ to the bilinear map $f = \otimes' \colon V \times W \to V \otimes_k' W (= Z')$, yielding a factorization $\tilde{f} \colon V \otimes_k W \to Z' = V \otimes_k' W$,
- of $V \otimes_k' W$ to the bilinear map $f' = \otimes \colon V \times W \to V \otimes_k W (= Z)$, yielding a factorization $\tilde{f}' \colon V \otimes_k W \to Z = V \otimes_k W$.

One uses the uniqueness of the factorization to show that $\tilde{f}' \circ \tilde{f} = \mathrm{id}_{V \otimes_k' W}$ and $\tilde{f} \circ \tilde{f}' = \mathrm{id}_{V \otimes_k W}$, hence $V \otimes_k W \cong V \otimes_k' W$.

- The *existence* of a tensor product is generally shown by constructing a free space and dividing out relations that force bilinearity. For vector spaces, one may simply choose bases $B_V = \{v_1, \dots\}$ of V and $B_W = \{w_1, \dots\}$ of W and consider the vector space $V \otimes_k W$ with basis elements $v_i \otimes w_j$. The bilinear map $\otimes \colon V \times W \to V \otimes_k W$ sends $(v, w) \mapsto v \otimes w$.
- Tensor products of morphisms: Let $\varphi \colon V \to V'$ and $\psi \colon W \to W'$ be morphisms of vector spaces. Then there is a natural tensor product of morphisms as well. $\varphi \otimes \psi \colon V \otimes W \to V' \otimes W'$ is the unique linear map with $(\varphi \otimes \psi)(v \otimes w) = \varphi(v) \otimes \psi(w)$.

Example. For an algebra A over a field k, the multiplication in A is by definition a *bilinear* map $A \times A \to A$.

Definition. For a vector space V over a field k, consider the map $c \colon V \otimes W \to V \otimes W$ given by $c(v \otimes w) = w \otimes v$ and define the subspaces

- $V \otimes_{\mathrm{sym}} V \coloneqq \{v \in V \otimes V \mid c(v) = v\} \subseteq V \otimes V$ (symmetric product),
- $V \wedge V \coloneqq \{v \in V \otimes V \mid c(v) = -v\} \subseteq V \otimes V$ (alternating product).

Alternatively, we may construct them as quotient spaces

- $V \otimes_{\mathrm{sym}} V = V \otimes V / \sim$ with $v \otimes w \sim w \otimes v$,
- $V \wedge V = V \otimes V / \sim$ with $v \otimes w \sim -w \otimes v$.

Example. For a vector space V, we define the vector space $TV = k \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots = \bigoplus_{n \geq 0} V^{\otimes n}$ and define a multiplication on a basis by

$$(v_1 \otimes \cdots \otimes v_k) \cdot (w_1 \otimes \cdots \otimes w_l) = v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_l.$$

Let $\{v_1, \dots, v_n\}$ be a basis of V, then TV is isomorphic to the free algebra in the n variables v_i. For $V = kX$, TV is the polynomial ring $TV \cong k[X]$.

Definition (Dual Space)**.** Let V be a vector space over k. The set of linear maps $V \to k$ with point-wise addition and multiplication forms a new vector space, the *dual space* V^*.

Remark. There is a natural linear map called *evaluation* $V^* \otimes V \to k, f \otimes v \mapsto f(v)$.

Definition (Inner Products)**.** For a real vector space V is a bilinear form $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ is called *inner product* if

- $\langle v, w \rangle = \langle w, v \rangle$ for all $v, w \in V$ (symmetric),
- $\langle v, v \rangle \geq 0$ for all $v \in V$ where $\langle v, v \rangle = 0 \implies v = 0$ (positive definiteness, in particular non-degenerate).

The pair $(V, \langle \cdot, \cdot \rangle)$ is called inner product space.

One can generalize this notion of the inner product for complex vector spaces to hermitian products by considering a sesquilinear form with $\langle v, w \rangle = \overline{\langle w, v \rangle}$ (and positive definiteness) instead.

Remark. We can view the inner product as a linear map $V \to V^*, v \mapsto (w \mapsto \langle v, w \rangle)$, which is injective (by the non-degeneracy of $\langle \cdot, \cdot \rangle$), and in finite dimensions even injective. Hence, an inner product corresponds to a choice of isomorphism $V \cong V^*$ (if the dimension is finite).

Definition. A basis $B = \{v_1, \ldots\}$ of a vector space is called *orthornormal basis* if $\langle v_i, v_j \rangle = \delta_{i,j}$ for all i, j.

Proposition. Given an orthonormal basis of a vector space, $v = \sum_i a_i v_i$ with $a_i = \langle v, v_i \rangle$.

Definition. For a linear endomorphism A of a real / complex inner product space $(V, \langle \cdot, \cdot \rangle)$ we define the *adjoint* endomorphism via $\langle v, Aw \rangle = \langle A^* v, w \rangle$ for all $v, w \in V$. A is called *self-adjoint* if $A^* = A$. (For a real vector space we call these operators *symmetric*, for a complex vector space we call them *hermitian.*)

Remark. For the representation matrix of A with respect to an orthonormal basis, taking the adjoint corresponds to taking the transpose and complex conjugate of the matrix.

Definition (Isometries)**.** A linear automorphism U of a real / complex vector space is called $(V, \langle \cdot, \cdot \rangle)$ *isometry* if $\langle Uv, Uw \rangle = \langle v, w \rangle$ for all $v, w \in V$. (This is the case if $U = U^*$.)

Note that it would suffice in finite dimensions to require U to be an endomorphism as the second condition would force it to be an automorphism as well. This is not true in general.
An isometry preserves the scalar product and hence lengths and angles. An isometry is called *unitary* for complex vector spaces and *orthogonal* for real vector spaces.

Example (Pauli Matrices)**.** Consider the vector space of self-adjoint, complex (2×2)-matrices. This space is 4-dimensional with basis

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

($\sigma_x, \sigma_y, \sigma_Z$ form a basis for the subspace of traceless self-adjoint matrices.)

Example. Rotations are orthogonal endomorphisms with determinant 1. Generally, an orthogonal map A (and its representing matrix with respect to an orthonormal basis) satisfies $A^\top = A^{-1}$ so that $\det(A) = \det(A^\top) = \det(A^{-1}) = \det(A)^{-1}$, hence $\det(A) \in \{\pm 1\}$. So, the isometry group of a real inner product space is generated by rotations and

$$\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

Example. In quantum mechanics, the complex self-adjoint endomorphisms (= hermitian operators) on the space of states $\mathcal{H}$ correspond to observables.

Examples.
- Hamilton operator / energy operator $\hat{H}$,
- position operator $\hat{x}$,
- momentum operator $\hat{p}$.

Example. Complex isometries (= unitary operators) of the state space $\mathcal{H}$ correspond to symmetries.
- the time evolution operator $U(t_2 - t_1)$,
- rotations, translations, ...
- discrete symmetries, parity reversal P, time reversal T or charge reversal C.

Definition (Eigenvalues)**.** For an endomorphism X of a vector space V over a field k, an *eigenvector* is a non-zero vector $v \in V$ such that

$$Xv = \lambda v$$

for some $\lambda \in k$. In that case, we call λ an *eigenvalue.*

We find the eigenvectors and eigenvalues of X in finite-dimensional vector spaces as follows:
- $Xv = \lambda v$ iff $(X - \lambda \cdot \mathrm{id})v = 0$,
- there exists a non-zero $v \in V$ such that $v \in \ker(X - \lambda\,\mathrm{id}_)$ (iff $\det(X - \lambda \cdot \mathrm{id})$).

- $\det(X - \lambda \cdot \mathrm{id})$ is called *characteristic polynomial* of X. Its roots / zeros are exactly the eigenvalues.
- For a given eigenvalue $\lambda \in k$, the eigenvectors are the kernel of $X - \lambda \cdot \mathrm{id}$.

Example. The matrix

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has characteristic polynomial

$$\det(X - \lambda \cdot \mathrm{id}) = \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1).$$

Hence, the eigenvalues are $\lambda_1 = 1, \lambda_2 = -1$, each with algebraic multiplicity 1. In particular, X is diagonalizable. Indeed, we find a basis of eigenvectors

$$\ker(X - \lambda_1 \cdot \mathrm{id}) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} k, \ker(X - \lambda_1 \cdot \mathrm{id}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} k.$$

Definition. Let X be an endomorphism of a finite dimensional vector space V over a field k.

- The *algebraic multiplicity* of an eigenvalue is the multiplicity of that eigenvalue as a root of the characteristic polynomial $\det(X - \lambda \cdot \mathrm{id})$.
- The *geometric multiplicity* is the dimension of the corresponding eigenspace $\ker(X - \lambda \cdot \mathrm{id})$.

Remark. If k is an algebraically closed field, every endomorphism has at least one eigenvalue and at least one eigenvalue.
In general, the geometric multiplicity can be smaller than the algebraic multiplicity. In this case, further elements are found in $\ker\left((X - \lambda \cdot \mathrm{id})^k\right), k \geq 1$.

Theorem. If k is an algebraically closed field and X is an endomorphism of a vector space V over k with algebraic multiplicity equalling the geometric ones for all eigenvalues, one can choose a basis of eigenvalues of X. The representing matrix of X is then diagonal with the eigenvalues as diagonal entries. In this case, we call X *diagonalizable.*

Theorem. More generally, over an algebraically closed field k, every endomorphism X admits a basis of V such that the representing matrix consists of Jordan blocks for each eigenvalue λ

$$\begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}.$$

The sum of dimensions of all blocks for an eigenvalue is its algebraic multiplicity, and the number of block is the geometric multiplicity.

Example. The Jordan block

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

has characteristic polynomial

$$\det(X - \lambda \cdot \mathrm{id}) = \det \begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^2.$$

Hence, the only eigenvalue is $\lambda = 1$ with algebraic multiplicity 2. We compute the eigenspace

$$\ker(X - \lambda \cdot \mathrm{id}) = \ker \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} k.$$

Hence, the geometric multiplicity is only 1 and X is not diagonalizable. However, we have as an "additional basis vector" in

$$\ker\left((X - \lambda \cdot \mathrm{id})^2\right) = \ker\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2\right) = \ker(0) = k^2.$$

The eigenvalue theory of self-adjoint endomorphisms $X^* = X$ on complex / real vector spaces with an inner product $\langle \cdot, \cdot \rangle$ is much stronger.

Easy facts:

- The eigenvalues of X are real because $Xv = \lambda v$ implies $Xv = X^*v = \overline{\lambda} v$ by assumption. Hence, $\lambda = \overline{\lambda}$.
- The eigenvalues for distinct eigenvalues $\lambda_1 \neq \lambda_2$ are orthogonal since

$$\lambda_1 \langle v, w \rangle = \langle Xv, w \rangle = \langle v, Xw \rangle = \lambda_2 \langle v, w \rangle \implies \langle v, w \rangle = 0.$$

Theorem. Let V be a finite-dimensional complex ($\leftarrow$ algebraically closed) inner product space and X a self-adjoint operator on X. Then X is diagonalizable and we can choose an orthonormal basis of eigenvectors of X.

Example. In quantum mechanics, there is a complex inner product space $\mathcal{H}$ and each observable is a self-adjoint endomorphism. The eigenvalues of X are the possible outcomes of a measurement of the observable and the expectation where for the measurement in a given state $\psi \in \mathcal{H}$ is

$$\langle X \rangle_\psi = \frac{\langle \psi, X\psi \rangle}{\langle \psi, \psi \rangle}.$$

2 Analysis & Geometry (Differential Geometry)

Definition. Let $p \in \mathbb{R}^n$. The *tangent space* at the point p is defined to be the set $T_p\mathbb{R}^n = \{(p, v) \mid v \in \mathbb{R}^n\}$, i.e. where we – morally speaking – "attach vectors to a point". (Vector space with addition / multiplication in the second component.)

The *differential* of a smooth (i.e. $\mathcal{C}^\infty$) map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ at the point p can be interpreted as a linear map between corresponding tangent spaces:

Definition. Let $U \subseteq \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}^m$ be a differentiable map. For each point p in U, the linear map $f_{*,p}\colon T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$ is defined by

$$f_{*,p}(p, v) = (f(p), Df_p(v)),$$

where

$$Df_p(v) = \lim_{t \to 0} \frac{f(p + tv) - f(p)}{t} = \left(\sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}\bigg|_p \right) f.$$

Definition (Vector Field)**.** A *vector field* on the open set $U \subseteq \mathbb{R}^n$ assigns a vector $\mathcal{V}(p) \in T_p\mathbb{R}^n$ in the corresponding tangent space to each point $p \in U$. If $\{e_1, \dots, e_n\}$ is the standard basis of Euclidean space, then the vector field determined by $p \mapsto (p, e_i) \in T_p\mathbb{R}^n$ is usually denoted $\frac{\partial}{\partial x_i}$, i.e. $\frac{\partial}{\partial x_i}(p) \coloneqq (p, e_i)$. Then every vector field can be written as

$$\mathcal{V}(p) = \sum_{i=1}^{n} V_i(p) \frac{\partial}{\partial x_i}(p)$$

with functions $V_1, \dots, V_n\colon U \to \mathbb{R}$.

Example. $\mathbb{R}^2$ with vector field $\mathcal{V}(x, y) = x\frac{\partial}{\partial y}$.

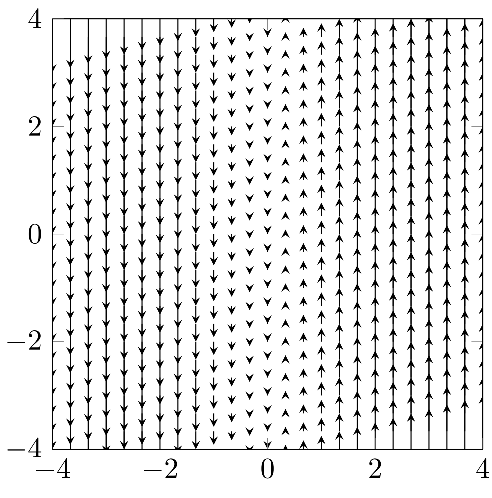


Figure 1: Plot of the vector field $\mathcal{V}(x, y)$

Each of the tangent spaces $T_p\mathbb{R}^n$ is a real vector space. Hence, we can take the *dual space* $T_p^*\mathbb{R}^n := (T_p\mathbb{R}^n)^*$ (called *cotangent space*) as well as the *exterior powers* $\bigwedge_p^k(\mathbb{R}^n) := \bigwedge^k(T_p^*\mathbb{R}^n)$. An element ω^k of the space $\bigwedge_p^k(\mathbb{R}^n)$ is thus an anti-symmetric, multilinear map with k arguments on the tangent space $T_p\mathbb{R}^n$:

$$\omega^k \colon T_p\mathbb{R}^n \times \cdots \times T_p\mathbb{R}^n \to \mathbb{R}.$$

Definition. A *(differential) k-form* on the open subset $U \subseteq \mathbb{R}^n$ assigns to each point $p \in U$ an element $\omega^k(p) \in \bigwedge_p^k(\mathbb{R}^n)$.

Example. Let $f \colon U \to \mathbb{R}$ be a smooth, real-valued function, let $p \in U$ be fixed, and let the directional derivative $Df_p \colon \mathbb{R}^n \to \mathbb{R}$ be its differential at the point p. Then the formula

$$df(p)(p,v) := Df_p(v)$$

defines a 1-form df on the set U. (At each point we have a linear map $\mathbb{R}^n \to \mathbb{R}$.)

Example. A fixed basis $\{e_1,\ldots,e_n\} \subseteq \mathbb{R}^n$ determines n coordinate functions $x_1,\ldots,x_n$ and hence their differentials $dx_1,\ldots,dx_n$. Thus, for a tangent vector $(p,v) \in T_p\mathbb{R}^n$ we have the identity

$$dx_i(p)(p,v) = v_i,$$

i.e. the i-th component of v with respect to $\{e_1,\ldots,e_n\}$. In addition, the 1-forms $dx_1(p),\ldots,dx_n(p)$ form a basis of the vector space $\bigwedge_p^1(\mathbb{R}^n) = T_p^*\mathbb{R}^n$.

Arbitrary exterior products $dx_{i_1} \wedge \cdots \wedge d_{x_{i_j}}$ as well as their linear combinations with functions as scalar coefficients lead to further examples of k-forms. Concretely, each k-form ω^k on U can be represented by

$$\omega^k = \sum_{i_1<i_2<\cdots<i_k} \omega_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Definition. A (possibly $l = \infty$) $\mathcal{C}^l$-differential form has coefficient functions in $\mathcal{C}^l(U)$. The set of all these forms is denoted by $\Omega_l^k(U)$. This is a real vector space, and it is also a module over the ring $\mathcal{C}^l(U)$.

Example. Let f be a real-valued function of class $\mathcal{C}^l$ on the open set $U \subseteq \mathbb{R}^n$. Then df is a 1-form of class $\mathcal{C}^{l-1}$ (in $\Omega_{l-1}^1(U)$):

$$df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n.$$

Indeed, at the point $p \in U$, the following holds for the vector (p,v):

$$df(p)(p,v) = Df_p(v) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p) v^i.$$

Replacing the vector components v^i by $dx_i(p)(p,v)$ and omitting the argument (p,v), we arrive at the stated formula:

$$df(p) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(p) dx_i(p).$$

We extend the exterior product of multilinear forms to differential forms, defining for two forms ω^k, η^l on U a $(k+l)$-form by

$$(\omega^k \wedge \eta^l)(p) \coloneqq \omega^k(p) \wedge \eta^l(p).$$

The same rules hold:

1. $(\omega^k + \mu^k) \wedge \eta^l = \omega^k \wedge \eta^l + \mu^k \wedge \eta^l$,
2. $(f \cdot \omega^k) \wedge \eta^l = f \cdot (\omega^k \wedge \eta^l) = \omega^k \wedge (f \cdot \eta^l)$,
3. $\omega^k \wedge \eta^l = (-1)^{kl} \eta^l \wedge \omega^k$.

In particular, the exterior product of a form of odd degree with itself is always 0, i.e. $dx_i \wedge dx_i = 0$.
Differential forms can be "pulled back" by maps.

Definition. Let $f \colon U_1 \to U_2$ be a differentiable map between two open subsets $U_1 \subseteq \mathbb{R}^n$ and $U_2 \subseteq \mathbb{R}^m$, and let ω^k be a k-form on U_2. Then a k-form $f^*(\omega^k)$ (the *pullback*) on U is defined by

$$f^*(\omega^k)\left((p, v_1), \ldots, (p, v_k)\right) \coloneqq \omega^k\left(f_{*,p}(p, v_1), \ldots, f_{*,p}(p, v_k)\right),$$

where $f_{*,p} \colon T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$.

Theorem. Let $f \colon U_1 \to U_2$ be a differentiable map between open sets $U_1 \subseteq \mathbb{R}^n$ and $U_2 \subseteq \mathbb{R}^m$ with component functions f_i. Then

$$f^*(dx_i) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} dx_j.$$

Moreover, for differentiable forms on U_2 and a function $g \colon U_2 \to \mathbb{R}$

1. $f^*(\omega_1^k + \omega_2^k) = f^*(\omega_1^k) + f^*(\omega_2^k)$,
2. $f^*(g \cdot \omega^k) = (g \circ f) \cdot f^*(\omega^k)$,
3. $f^*(\omega^k \wedge \eta^l) = f^*(\omega^k) \wedge f^*(\eta^l)$.

We now view $\mathcal{C}^l$-functions f on U as 0-forms, so that the differential d can be viewed as a map turning a 0-form into a 1-form df. Applying this map in a suitable way to the coefficient function of a differential form, the exterior derivative can be extended to act on k-forms:

Definition. Let ω^k be a k-form on the open set U

$$\omega^k = \sum_{i_1 < \cdots < i_k} \omega_{i_1, \ldots, i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

We define the exterior derivative

$$d\omega^k = \sum_{i_1 < \cdots < i_k} d(\omega_{i_1, \ldots, i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$$= \sum_{i_1<\cdots<i_k} \sum_{\alpha=1}^{n} \frac{\partial \omega_{i_1,\ldots,i_k}}{\partial x_\alpha} dx_\alpha \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

d becomes a linear operator

$$d\colon \Omega_l^k(U) \to \Omega_{l-1}^{k+1}(U).$$

Example. Consider on $\mathbb{R}^2$ the 1-form

$$\omega^1 = \sin(x)dy + \sin(y)dx.$$

Since

$$\begin{aligned} d(\sin(x)) &= \cos(x)dx + 0 \cdot dy \\ d(\sin(y)) &= 0 \cdot dx + \cos(y)dy, \end{aligned}$$

we have

$$\begin{aligned} d\omega^1 &= \cos(x)dx \wedge dy + \cos(y)dy \wedge dx \\ &= (\cos(x) - \cos(y))dx \wedge dy. \end{aligned}$$

Theorem. The exterior derivative obeys:

1. $d(\omega^k + \eta^k) = d\omega^k + d\eta^k, \omega^k, \eta^k \in \Omega_1^k(U)$,
2. $d(\omega^k \wedge \eta^l) = (d\omega^k) \wedge \eta^l + (-1)^k \omega^k \wedge (d\eta^l)$,
3. $d(d\omega^k) = 0$ for $\omega^k \in \Omega_2^k(U)$,
4. $f^*(d\omega^k) = d(f^*\omega^k)$ for $\omega^k \in \Omega_1^k(U_2), f\colon U_1 \to U_2 \subseteq \mathbb{R}^m$.

Example. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^3, f(u,v) = (u^2, v^3, uv)$ and the 1-form on $\mathbb{R}^3$ defined in the coordinates x, y, z by

$$\omega^1 = ydx + xdy + xyzdz.$$

We compute $f^*(\omega^1)$ as follows

$$\begin{aligned} f^*(\omega^1) &= (y \circ f)f^*(dx) + (x \circ f)f^*(dy) + (xyz \circ f)f^*(dz) \\ &= v^3 f^*(dx) + u^2 f^*(dy) + u^3 v^4 f^*(dz). \end{aligned}$$

Then we use the fact that the exterior derivative commutes with the pullback, so that

$$\begin{aligned} f^*(dx) &= d(f^*x) = d(x \circ f) = d(u^2) = 2udu \\ f^*(dy) &= 3v^2 dv \\ f^*(dz) &= vdu + udv, \end{aligned}$$

so that

$$f^*(\omega^1) = (2uv^3 + u^3v^5)du + (3u^2v^2 + u^4v^4)dv.$$

Recall: Every continuous function on $\mathbb{R}$ has a primitive function. In other words, for every 1-form $\mu^1 = g(x)dx$ with a continuous coefficient function $g\colon \mathbb{R} \to \mathbb{R}$, there exists a function f such that $df = \mu^1$. If in addition g is differentiable, then certainly $d\mu^1 = 0$ since each 2-form on $\mathbb{R}$ vanishes.

Definition.

1. A k-form $\omega^k \in \Omega_1^k(U)$ is called *closed* if $d\omega^k = 0$.
2. A k-form $\omega^k \in \Omega_1^k(U)$ is called *exact* if there exists a $(k-1)$-form $\eta^{k-1} \in \Omega_2^{k-1}(U)$ such that $d\eta^{k-1} = \omega^k$.

Remark. The property $dd = 0$ states that each exact form is closed. The converse is not true.

Example. Consider on the open set $\mathbb{R}^3 \setminus \{0\}$ the winding form

$$\omega^1 = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

We can calculate its derivative $d\omega^1 = 0$. Hence, ω^1 is closed but it's *not* exact.

Definition. We define the vector space of "cycles"

$$Z^k(U) := \Big\{\omega^k \in \Omega_\infty^k(U) \mid \omega^k \text{ is closed.}\Big\}$$

and of the "boundaries"

$$B^k(U) := \Big\{\omega^k \in \Omega_\infty^k(U) \mid \omega^k \text{ is exact.}\Big\}$$

and the *k-th de Rham cohomology*

$$H_{dR}^k(U) = Z^k(U)/B^k(U).$$

Example. The winding form is a non-trivial element in $H_{dR}^1(\mathbb{R}^2 \setminus \{0\}) \neq \{0\}$. In fact, $H_{dR}^1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{R}$ (as vector space) ($\iff$ $\dim(H_{dR}^1(\mathbb{R}^2 \setminus \{0\})) = 1$). The k-th de Rham cohomology only depends on the topological shape of U. For instance:

Theorem (Poincaré's lemma)**.** Let U for a star-shaped open set in $\mathbb{R}^n$. Then

$$H_{dR}^k(U) = \{0\}$$

for every $k = 1, \ldots, n$. In other words, for each k-form $\omega^k \in \Omega_1^k(U)$ there exists a $(k-1)$-form $\eta^{k-1} \in \Omega_2^{k-1}(U)$ such that $d\eta^{k-1} = \omega^k$.

Example. Consider on $\mathbb{R}^3$ the closed 2-form

$$\omega^2 = xydx \wedge dy + 2xdy \wedge dz + 2ydx \wedge dz.$$

We will determine a 1-form whose derivative coincides with ω^2.

Ansatz: The 1-form $\eta^1 = f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz$ for some $f,g,h \in \mathbb{R}^3 \to \mathbb{R}$. The exterior derivative is

$$d\eta^1 = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy + \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right) dx \wedge dz + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy \wedge dz.$$

Hence, the functions have to satisfy the conditions

$$\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} = xy, \frac{\partial h}{\partial x} - \frac{\partial f}{\partial z} = 2y \text{ and } \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} = 2x.$$

Integrating, e.g., the first two with respect to x gives us

$$g = \frac{1}{2}x^2 y + \int \frac{\partial f}{\partial y} \, \mathrm{d}\, x, h = 2xy + \int \frac{\partial f}{\partial z} \, \mathrm{d}\, x.$$

Inserting the result into the last condition

$$2x = 2x + \int \frac{\partial^2 f}{\partial y \partial z} \, \mathrm{d}\, x - \int \frac{\partial^2 f}{\partial z \partial y} \, \mathrm{d}\, x,$$

we see that any function f satisfies this, in particular $f = 0$. Then, $g = x^2 y/2, h = 2xy$ and hence

$$\eta^1 = \frac{x^2 y}{2} dy + 2xy dz.$$

Each tangent space $T_p\mathbb{R}^n$ is an oriented, Euclidean vector space, and hence this is the volume form

$$d\mathbb{R}^n(p) \in \bigwedge_p^n (\mathbb{R}^n)$$

($d\mathbb{R}^n$ is just $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$.). We also have the *Hodge operator*

$$*\colon \bigwedge_p^k (\mathbb{R}^n) \to \Omega_p^{n-k}(\mathbb{R}^n) \tag{1}$$

which is uniquely determined by the condition that for any positively oriented orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ and dual basis $\{e^1, \ldots, e^n\} \subseteq (\mathbb{R}^n)^*$ we have

$$*(e^1 \wedge \cdots \wedge e^k) = e^{k+1} \wedge \cdots \wedge e^n.$$

We associate with every differential form ω^k of degree k on $\mathbb{R}^n$ a corresponding $(n-k)$-form $*\omega^k$ defined by applying the $*$-operator pointwise. Consider an orthonormal basis $\{e_1, \ldots, e_n\} \subseteq \mathbb{R}^n$ and corresponding coordinate functions $x_1, \ldots, x_n$. For a k-form

$$\omega^k = \sum_{I \text{ multi-index}} \omega_I dx^I$$

the associated form is

$$*\omega^k = \sum_I \underbrace{\operatorname{sgn}\begin{pmatrix} 1 \ldots n \\ I, j \end{pmatrix}}_{\in\{\pm 1\}} \omega_I dx^I.$$

Here $J = (j_1 < \cdots < j_{n-k})$ is the complementary multi-index for $I = (i_1 < \cdots < i_k)$ and

$$\begin{pmatrix} 1 \ldots n \\ I, J \end{pmatrix}$$

is the permutation mapping $1 \mapsto i_1, \ldots, k \mapsto i_k, k+1 \mapsto j_1, \ldots, n \mapsto j_n$.
We can pass from vector fields to a 1-form and vice versa.

Definition. For a vector field $\mathcal{V}$ the dual 1-form $\omega_{\mathcal{V}}$ is defined by $*\omega^1_{\mathcal{V}} := \mathcal{V} \lrcorner d\mathbb{R}^n$, where at every point, $\lrcorner$ takes the k-form to a $(k-1)$-form by inserting a vector in the first argument of the k-form. If the vector field

$$\mathcal{V} = \sum_{i=1}^{n} V_i \frac{\partial}{\partial x_i}$$

is expressed in Cartesian coordinates, then

$$\omega^1_{\mathcal{V}} = V_1 dx_1 + \cdots + V_n dx_n.$$

Definition.
- The *divergence* of a $\mathcal{C}^1$-vector field is the function determined by the following equation
$$d(*\omega^1_{\mathcal{V}}) = d(\mathcal{V} \lrcorner d\mathbb{R}^n) := \operatorname{div}(\mathcal{V}) \cdot d\mathbb{R}^n.$$
The formula $\operatorname{div}(\mathcal{V}) = \sum_{i=1}^n \frac{\partial V_i}{\partial x_i}$ expresses the divergence of $\mathcal{V}$ through its components.
- Let $f\colon U \to \mathbb{R}$ be a $\mathcal{C}^1$-function defined on an open subset $U \subseteq \mathbb{R}^n$. The *gradient* $\operatorname{grad}(f)$ is the vector field
$$*\omega^1_{\operatorname{grad}(f)} = \operatorname{grad}(f) \lrcorner d\mathbb{R}^n = *df.$$
In the chosen coordinates we have
$$\operatorname{grad}(f) = \frac{\partial f}{\partial x_1} \frac{\partial}{\partial x_1} + \cdots + \frac{\partial f}{\partial x_n} \frac{\partial}{\partial x_n}.$$

Theorem.
1. Let f, g be $\mathcal{C}^1$-functions. Then
$$\operatorname{grad}(f \cdot g) = f \cdot \operatorname{grad}(g) + g \cdot \operatorname{grad}(f).$$
2. For a function f and a vector field $\mathcal{V}$ in $\mathcal{C}^1$
$$\operatorname{div}(f \cdot \mathcal{V}) = f \cdot \operatorname{div}(\mathcal{V}) + df(\mathcal{V}).$$

Definition. Let f be a $\mathcal{C}^1$-function defined on an open subset $U \subseteq \mathbb{R}^n$. The *Laplacian* is $\Delta(f) = \operatorname{div}(\operatorname{grad}(f))$. In the chosen coordinates the Laplacian is

$$\Delta(f) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}.$$

In dimension 3 there exists an additional operation acting on vector fields.

Definition. The *curl* of the vector field $\mathcal{V}$ is the unique vector field defined by

$$d\omega^1_{\mathcal{V}} =: \mathrm{curl}(\mathcal{V}) \lrcorner d\mathbb{R}^3 = *\omega^1_{\mathrm{curl}(\mathcal{V})}.$$

It satisfies

$$\mathrm{curl}(\mathcal{V}) = \left(\frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3}\right) \cdot \frac{\partial}{\partial x_1} + \left(\frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1}\right) \cdot \frac{\partial}{\partial x_2} + \left(\frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right) \cdot \frac{\partial}{\partial x_3}.$$

Theorem. If the function f and the vector field $\mathcal{V}$ are in $\mathcal{C}^2$ on $U \subseteq \mathbb{R}^3$, then:

1. $\mathrm{div}(\mathrm{curl}(\mathcal{V})) = 0$ and $\mathrm{curl}(\mathrm{grad}(f)) = 0$ ($dd = 0!$),
2. $\mathrm{curl}(f \cdot \mathcal{V}) = f \cdot \mathrm{curl}(\mathcal{V}) + \mathrm{grad}(f) \times \mathcal{V}$, where $\times$ is the $\mathbb{R}^3$-cross / vector product,
3. if $\mathrm{curl}(\mathcal{V}) = 0$ and U is star-shaped, then there is a function f such that $\mathcal{V} = \mathrm{grad}(f)$,
4. if $\mathrm{div}(\mathcal{V}) = 0$ and U is star-shaped, then there is a vector field $\mathcal{W}$ such that $\mathcal{V} = \mathrm{curl}(\mathcal{W})$.

The theorem is a direct consequence of Poincaré's lemma.
In the following we shall want to integrate differential forms. First, we denote suitable sets as integration domains.

Definition. A *singular k-cube* in $U \subseteq \mathbb{R}^n$ is a $\mathcal{C}^1$-map $c^k \colon [0,1]^k \to U$ with the unit cube as its domain ($k = 0$: point, $k = 1$: curve).

Definition. A *singular k-chain* in U is a formal sum of singular k-cubes c^k_i in U with integer coefficients $l_i \in \mathbb{Z}$

$$l_1 c^k_1 + \cdots + l_n c^k_n =: s^k.$$

This forms the abelian group $C_k(U)$ of singular k-chains.

Definition. The *standard cube* in $\mathbb{R}^k$ is defined to be the identity map of the k-dimensional unit cube

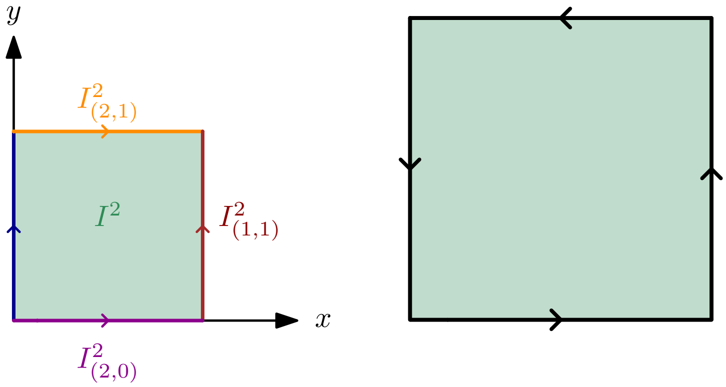
$$I^k \colon [0,1]^k \to \mathbb{R}^k, I^k(x) = x.$$

Definition. Let $I^k_{(i,0)}$ and $I^k_{(i,1)} \colon [0,1]^{k-1} \to [0,1]^k \subseteq \mathbb{R}^k$ with

$$I^k_{(i,0)} = (x_1, \dots, x_{i-1}, 0, x_i, \dots, i_{k-1}) \text{ and } I^k_{(i,1)} = (x_1, \dots, x_{i-1}, 1, x_i, \dots, i_{k-1}).$$

The *boundary* of the k-standard cube is then taken to be the $(k-1)$-chains

$$\partial I^k := \sum_{i=1}^{k} (-1)^i \left(I^k_{(i,0)} - I^k_{(i,1)}\right).$$



(a) Boundary for $k = 2$ (b) With sign

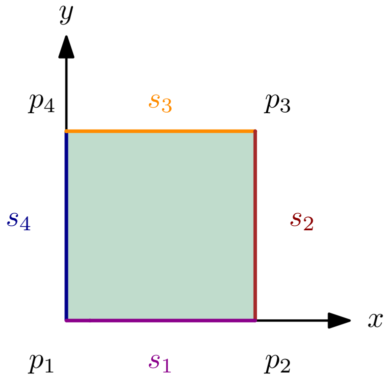
For an arbitrary singular k-cube $c^k\colon [0,1]^k \to U \subseteq \mathbb{R}^n$ we define

$$\partial c^k = \sum_{i=1}^{k} (-1)^i \left(c^k \circ I^k_{(i,0)} - c^k \circ I^k_{(i,1)} \right).$$

Finally, for a singular k-chain $s^k = \sum_j l_j c^k_j$ we define $\partial s^k = \sum_j l_j \partial c^k_j$.

Example. We compute the boundary of I^2. Then

$$\begin{aligned} \partial I^2 &= s_1 + s_2 + s_3 + s_4 \\ \partial\partial I &= (p_2 - p_1) + (p_3 - p_2) + (p_4 - p_3) + (p_1 - p_4) = 0. \end{aligned}$$



Theorem.

1. The boundary operator $\partial\colon C_k(U) \to C_{k-1}(I)$ is a group homomorpism.
2. $\partial\partial = 0$, so for every k-chain $s^k \in C_k(U)$ we have $\partial(\partial s^k) = 0$.

Definition. Similarly to the de Rham cohomology, we define the *k-th cube homology group*

$$H_k^{cub}(U) := \frac{\ker\left(\partial\colon C_k(U) \to C_{k-1}(U)\right)}{\operatorname{im}\left(\partial\colon C_{k+1}(U) \to C_k(U)\right)}.$$

Consider a singular k-cube $c^k\colon [0,1]^k \to U \subseteq \mathbb{R}^n$ in $\mathcal{C}^1$ and a k-form ω^k on U. Then the induced differential form $\left(\mathcal{C}^k\right)^* \omega^k$ is defined on the unit cube $[0,1]^k$, and as such a multiple of $f(x)dx_1 \wedge \cdots \wedge dx_k$ for some function $f\colon [0,1]^k \to \mathbb{R}$

Definition. Set

$$\int_{c^k} \omega^k := \int_{[0,1]^k} f(x)$$

and extend this linearly to k-chains $s^k = \sum_j l_j c_j^k$ in U by

$$\int_{s^k} \omega^k = \sum_j l_j \int_{c_j^k} \omega^k.$$

Example. For $k = 1$, a singular k-cube is a parameterized $\mathcal{C}^1$-curve $c\colon [0,1] \to U \subseteq \mathbb{R}^n$. In this case, the integral of a 1-form $\omega^1 = p_1 dy_1 + \cdots + p_n dy_n$ is called the *line integral* of ω^1 along c. If $c_1, \ldots, c_n$ are the component functions of c, then the pullback of the form is

$$c^*\omega^1(t) = p_1(c(t)) \cdot \frac{dc_1(t)}{dt} dt + \cdots + p_n(c_t) \cdot \frac{dc_n(t)}{dt} dt$$

(1-form on $[0,1] \subseteq \mathbb{R}$) and hence, in this situation, we obtain the following general formula for the line integral:

$$\int_c \omega^1 = \int_0^1 \sum_{i=1}^n p(c_i(t)) \cdot \frac{\mathrm{d}\, c_i(t)}{\mathrm{d}\, t} \,\mathrm{d}\, t.$$

If the 1-form ω^1 is the differential of a smooth function f, then by $c^*(\omega) = c^*(df) = d(f \circ c)$ we obtain

$$\int_c df = \int_0^1 \frac{\mathrm{d}(f \circ c)(t)}{\mathrm{d}\, t} \,\mathrm{d}\, t = f(c(1)) - f(c(0)).$$

The line integral of an *exact* 1-form depends only on the endpoints of the curve and not on its precise shape. It vanishes over a closed curve.

Theorem (Stokes' Theorem)**.** Let ω^k be a differential form on the open subset $U \subseteq \mathbb{R}^n$, and $s^{k+1}\colon [0,1]^k \to U$ be a $(k+1)$-chains. Then

$$\int_{\partial s^{k+1}} \omega^k = \int_{s^{k+1}} \mathrm{d}\,\omega^k.$$

("For exact forms, the boundary is the only thing that matters.")

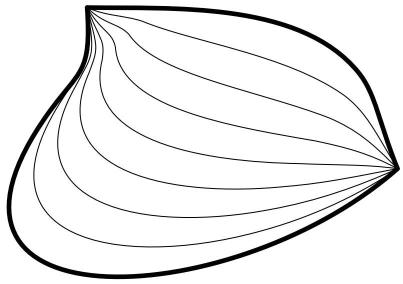


Figure 2: Homotopy between two paths (Source: Wikimedia)

Theorem. Let $c_0, c_1 \colon [0,1] \to U$ be the *homotopic* $\mathcal{C}^1$-curves, and ω^1 be a closed 1-form on U. Then

$$\int_{c_1} \omega^1 = \int_{c_2} \omega^1.$$

For simplicity, we restrict to $\mathcal{C}^\infty$ (*smooth*) *sub*manifolds of $\mathbb{R}^n$. (There is a notion of manifolds independent of an embedding.)

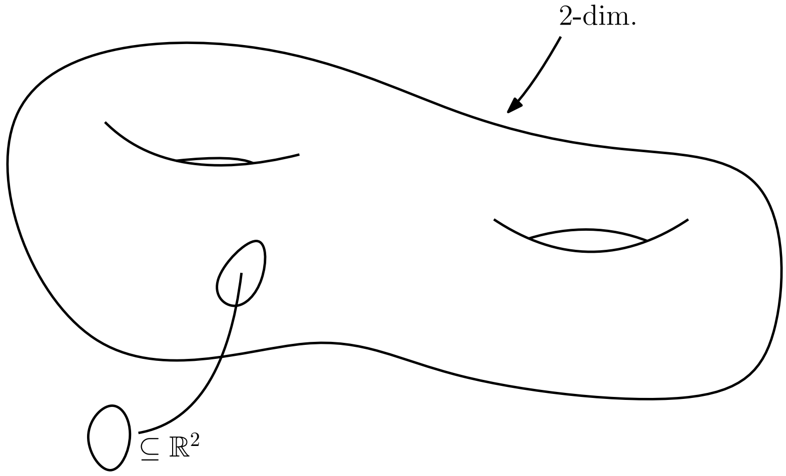


Figure 3: 2-dimensional submanifold in $\mathbb{R}^3$

Definition. A k-dimensional (smooth) *submanifold* M (without boundary) in $\mathbb{R}^n$ is a subset $M \subseteq \mathbb{R}^n$ such that for every point $x \in M$, there is an open neighborhood $U \subseteq \mathbb{R}^n$, such that we can find an open set $V \subseteq \mathbb{R}^n$ and a diffeomorphism (i.e. a smooth bijection with smooth inverse) $h \colon U \to V$ with

$$h(U \cap M) = V \cap \left(\mathbb{R}^k \times \{0\}^{n-k}\right) \subseteq \mathbb{R}^n.$$

The neighborhood $U^* = U \cap M$ with the diffeomorphism

$$h^* = h|_{U^*} \colon U^* \to V^* \left(:= V \cap \left(\mathbb{R}^k \times \{0\}^{n-k}\right)\right)$$

is called *chart*. For two charts $(h_1^*, U_1^*), (h_2^*, U_2^*)$ for which for which $U_1^* \cap U_2^* \neq \emptyset$, there is a transition map

$$h_2^* \circ h_1^{*-1} \colon \underbrace{h_2\left(U_1^* \cap U_2^*\right)}_{\text{open in } \mathbb{R}^k} \to \underbrace{h_1\left(U_1^* \cap U_2^*\right)}_{\text{open in } \mathbb{R}^k}$$

which is a diffeomorphism.

Submanifolds can be defined by equations:

Theorem. Let $U \subseteq \mathbb{R}^n$ be open and $f = (f_1, \ldots, f_{n-k}) \colon U \to \mathbb{R}^{n-k}$ be a smooth map. Consider the set

$$M = \{x \in U \mid f(x) = 0\}$$

If the differential $Df(x)$ has (maximal) rank $n - k$ at each point in M, then M is a smooth k-dimensional submanifold of $\mathbb{R}^n$.

For the proof of this, use the implicit function theorem.

Example. Every open subset $U \subseteq \mathbb{R}^n$ is an n-dimensional submanifold.

Example. The sphere $S^n = \left\{x \in \mathbb{R}^{n+1} \mid \|x\|^2 = 1\right\}$ is an n-dimensional submanifold.

To see this, we consider the function $f \colon \mathbb{R}^{n+1} \to \mathbb{R}$

$$f(x) = \|x\|^2 + 1 = x_1^2 + \cdots + x_{n+1}^2 - 1.$$

Then $\partial f / \partial x_i = 2x_i$ and the rank of the rank of the $(1 \times (n+1))$-matrix $(\partial f / \partial x_i)_{i=1,\ldots,n+1}$ on S^n is equal to 1.

Theorem. Let M be a subset of $\mathbb{R}^n$ and assume that for each point $x \in M$ there are an open set $U \subseteq \mathbb{R}^n$, an open set $W \subseteq \mathbb{R}^k$ and a smooth map $f \colon W \to U$ such that

1. $f(W) = M \cap U$,
2. f is bijective,
3. the differential Df has rank k at each point $y \in W$,
4. $f^{-1} \colon M \cap U \to W$ is continuous.

Then M is a k-dimensional submanifold of $\mathbb{R}^n$.

We extend the notion of submanifolds to also allow boundaries. We define the k-*dimensional half-space* as

$$\mathbb{H}^k \coloneqq \{x \in \mathbb{R}^n \mid x_k \geq 0\}.$$

Definition. A subset $M \subseteq \mathbb{R}^n$ is called a k*-dimensional sumanifold (with boundary)* if for each poitn $x \in M$ one of the following holds:

1. There exists an open set U and V with $x \in U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^n$ and a diffeomorphism $h \colon U \to V$ such that

$$h(U \cap M) = V \cap \left(\mathbb{R}^k \times \{0\}^{n-k}\right).$$

2. There exists open sets U and V with $x \in U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^n$ and a diffeomorphism $h\colon U \to V$ such that

$$h(U \cap M) = V \cap \left(\mathbb{H}^k \times \{0\}^{n-k}\right)$$

and the k-th component h_k of h vanishes at the point x, i.e. $h_k(x) = 0$.

The points $x \in M$ where 2. is satisfied are called *boundary* of M and denoted by ∂M.

Theorem. Let U be a k-dimensional manifold with boundary ∂M is either empty or a smooth $(k-1)$-dimensional manifold without boundary. Indeed, $\partial\partial M = \emptyset$.

Example. The $(n-1)$-dimensional sphere $S^{n-1} \subseteq \mathbb{R}^n$ is the boundary of the n-dimensional ball $\mathbb{D}^n$.

In the following, let M^k be a k-dimensional submanifold and let $h\colon U \to V$ be a chart around some point x and denote $y := h(x)$ Then h^{-1} is smooth and $(Dh^{-1})_y = (h^{-1})_{*,y}$ a linear map between tangent spaces

$$(h^{-1})_{*,y}\colon T_y\mathbb{R}^n \to T_x\mathbb{R}^n.$$

Definition. The *tangent space* of the manifold M^k at the point x is defined as

$$T_xM^k := (h^{-1})_{*,y}(T_y\mathbb{R}^n) \subseteq T_x\mathbb{R}^n.$$

The tangent space T_xM^k is a k-dimensional vector space because h^{-1} is injective. (We denote by the tangent bundle TM^k the set of all tangent spaces.)

Theorem. The tangent space T_xM^k consists of all vectors $(x, v) \in T_x\mathbb{R}^n$ for which there exists a smooth curve $\gamma\colon (-\varepsilon, \varepsilon) \to M^k$ such that $\gamma(0) = x, \gamma'(0) = v$.

Theorem. Let $f_1, \ldots, f_{n-k}\colon \mathbb{R}^n \to \mathbb{R}$ be smooth functions and suppose hat $df_1 \wedge \cdots \wedge df_{n-k} \neq 0$. Then the tangent space T_xM^k of the manifold

$$M^k = \{x \in \mathbb{R}^n \mid f_1(x) = \cdots = f_{n-k}(x) = 0\}$$

consists of all vectors $v \in T_x\mathbb{R}^n$ satisfying

$$df_1(v) = \cdots = df_{n-k}(v) = 0.$$

In particular, the Euclidean gradient fields $\operatorname{grad}(f_1), \ldots, \operatorname{grad}(f_{n-k})$ are perpendicular to the tangent space of the manifold at each point of M^k.

Example. Consider the n-sphere $S^n = \left\{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\right\}$. The differential of the function $\|x\|^2 - 1 = x_1^2 + \cdots + x_{n+1}^2 - 1$ is $2x_1dx_1 + \ldots 2x_{n+1}dx_{n+1}$ $((2x_1, \ldots, 2x_{n+1}))$ and hence the tangent space to the sphere at any point consists of all the vectors perpendicular to this point:

$$TS^n = \left\{(x, v) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|x\| = 1, \langle x, v\rangle = 0\right\}$$

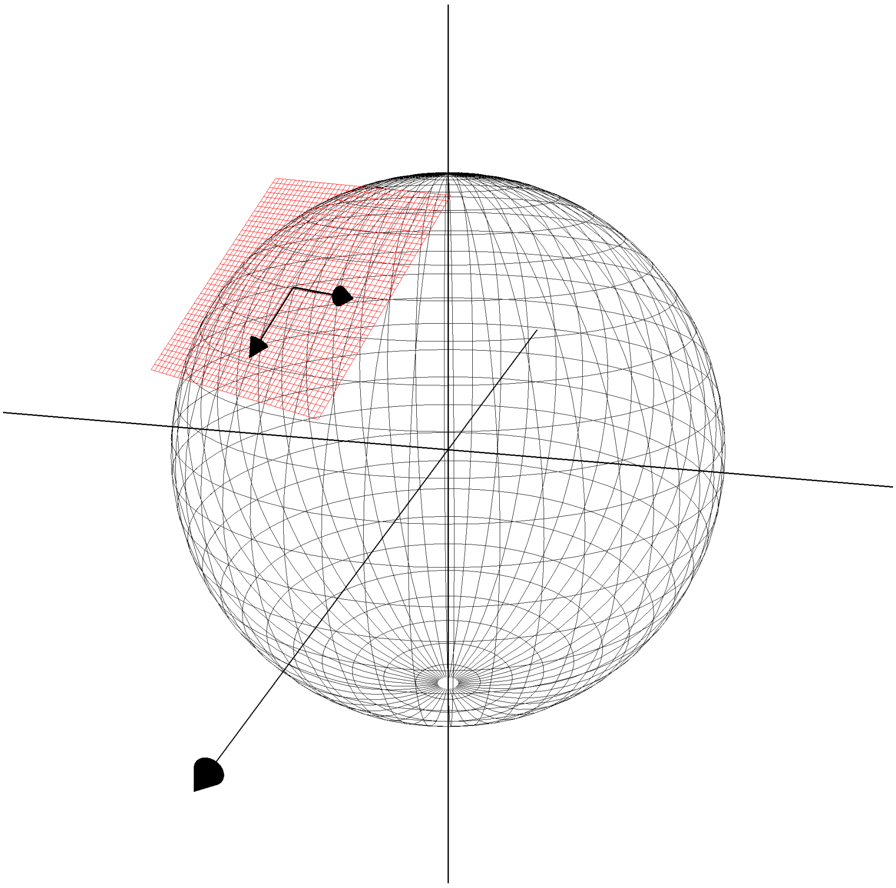


Figure 4: Tangent space of S^2 (Source: trecs.se)

Definition. Let $M^k \subseteq \mathbb{R}^n$ and $N^l \subseteq \mathbb{R}^m$ be two manifolds and let $f\colon M^k \to N^l$ b a continuous map. We call f *differentiable* if for each chart $h^{-1}\colon V \to M^k$ of the manifold M^k the resulting map $f \circ h^{-1}\colon V \to N^l \subseteq \mathbb{R}^m$ defined on the open subset $V \subseteq \mathbb{R}^k$ is differentiable.

Definition. A *vector field* $\mathcal{V}$ on a manifold M^k assigns to every point $x \in M^k$ a vector $\mathcal{V}(x) \in T_x M^k$ in the corresponding tangent space. (We can again multiply by functions so that, e.g., the smooth vector fields form a module over the ring $\mathcal{C}^\infty(M^k)$ of smooth functions on M^k.)

Example. The formula $\mathcal{V}(x) = (x, (x_2, -x_1, 0))$ defines a vector field on the 2-dimensional sphere (embedded in $\mathbb{R}^3$).

Notation. For a chart map $h\colon V \to M^k \subseteq \mathbb{R}^n$ (here and sometimes below we call it h rather than h^{-1}), $h_x(\partial/\partial y_i)$ are vector fields tangent to M^k defined on the subset $h(V) \subseteq M^k$, and they provide a basis in each tangent space. For simplicity, we also

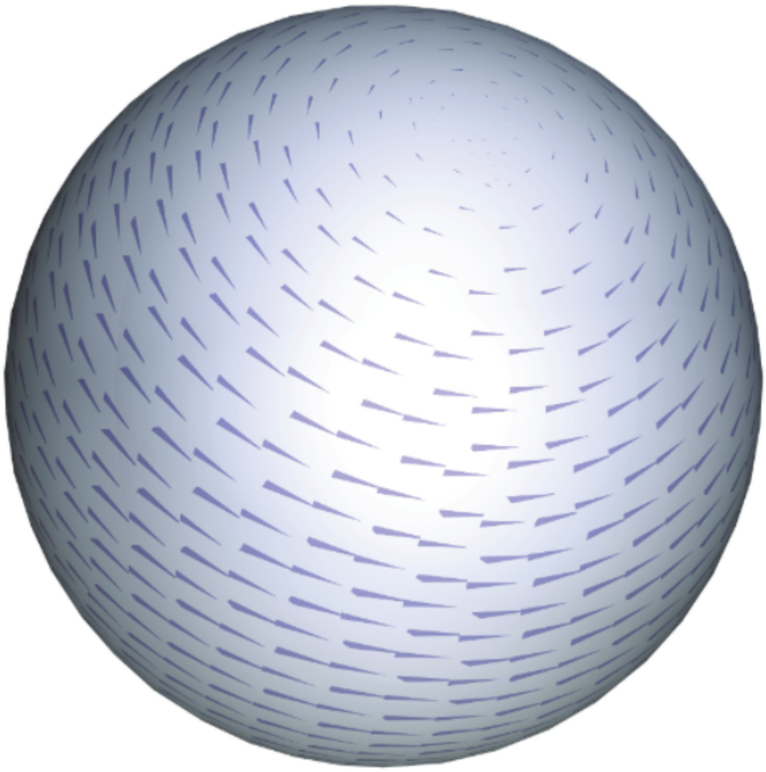


Figure 5: $\mathcal{V}(x) = (x, (x_2, -x_1, 0))$
(Out of Ben-Chen et al., "On Discrete Killing Vector Fields and Patterns on Surfaces")

denote these by $\partial/\partial y_i$. On $h(V) \subseteq M^k$, every vector field $\mathcal{V}$ can be written as

$$\mathcal{V}(g) = \sum_{i=1}^{k} V_i(g) \frac{\partial}{\partial y_i},$$

where the V_i's are functions on $h(V)$ (using the chart map we sometimes consider them functions on V, the parameter set).

Example. In Euclidean coordinates consider the vector field

$$\mathcal{V} = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}.$$

Introducing in $\mathbb{R}^3 \setminus \{0\}$ polar coordinates

$$h(r, \varphi) = (r \cos \varphi, r \sin \varphi), 0 < r < \infty, 0 \leq \varphi \leq 2\pi,$$

we see that $\mathcal{V}$ corresponds to the vector field $\partial/\partial\varphi$.

Differentiable funcitons $f\colon M^k \to \mathbb{R}$ can be differentiated with respect to a vector field $\mathcal{V}$. At a fixed point $x \in M^k$ we choose a curve $\gamma\colon [0,\varepsilon] \to M^k$ satisfying the initial conditions $\gamma(0) = x$ and $\dot\gamma(0) = \mathcal{V}(x)$. The *derivative* of f at the point x in the direction $\mathcal{V}(x)$ is now defined by the formula

$$\mathcal{V}(f)(x) \coloneqq \frac{\mathrm{d}}{\mathrm{d}\,t} f \circ \gamma(t)\Big|_{t=0}.$$

The result is a $\mathcal{C}^\infty$-function $\mathcal{V}(f)$ defined on the manifold M^k. If

$$\mathcal{V} = \sum_{i=1}^{k} V_i(y) \frac{\partial(f \circ h)}{\partial y_i}.$$

Let $\langle \cdot, \cdot \rangle$ denote the scalar product in $\mathbb{R}^n$. We restrict it to the tangent spaces of the submanifold.

Definition. Let $M^k \subseteq \mathbb{R}^n$ be a submanifold. In each tangent space $T_x M^k$ the formula

$$g_x\left((x,v),(x,w)\right) \coloneqq \langle v, w \rangle$$

defines a scalar product. The family $\{g_x\}$ of all these scalar products is called the *Riemannian metric* of M^k. In a chart $h\colon V \to M^k$ the Riemannian metric is locally described by the functions $g_{i,j}$ defined on the set V,

$$g_{i,j}(y) = g_{h(g)}\left(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right) = \left\langle \frac{\partial h}{\partial y_i}, \frac{\partial j}{\partial y_j} \right\rangle.$$

The $k \times k$-matrix $(g_{i,j})_{1 \le i,j \le k}$ is symmetric and positive-definite for each $y \in V$, We also define

$$g(y) \coloneqq \det\left((g_{i,j}(y))\right)$$

and denote by $(g^{i,j}(y))$ the inverse matrix of $(g_{i,j}(y))$.

Example. In Euclidean coordinates on $\mathbb{R}^n$, the entries $g_{i,j}(y) = \delta_{i,j}$ are constant.

Example. In polar coordinates on $\mathbb{R}^2 \setminus \{0\}$ we have

$$g_{r,r} = \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle = 1, g_{r,\varphi} = \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \varphi} \right\rangle = 0, g_{\varphi,\varphi} = \left\langle \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \varphi} \right\rangle = r^2.$$

In particular, $g(r,\varphi) = r^2$.

We can use the Riemannian metric g of a manifold to associate with every smooth function a vector field, the gradient.

Definition. For a fixed point $x \in M^k$ and a tangent vector $v \in T_x M^k$, we first choose a vector field $\mathcal{V}$ such that $\mathcal{V}(x) = v$. The assignment $v \mapsto \mathcal{V}(f)(x)$ determines a linear functional $T_xM^k \to \mathbb{R}$ on the tangent space, and hence there exists a vector $\mathrm{grad}(f)(x) \in T_xM^k$ such that

$$\mathcal{V}(f)(x) = g_x\left(\mathrm{grad}(f)(x), \mathcal{V}(x)\right)$$

holds for all vector fields. The vector field $\mathrm{grad}(f)$ is called the *gradient* of the smooth function $f\colon M^k \to \mathbb{R}$. If $h\colon V \to M^k$ is a chart, then

$$\mathrm{grad}(f) = \sum_{i,j=1}^{k} \frac{\partial(f \circ h)}{\partial y_i} g^{i,j} \frac{\partial}{\partial y_j}.$$

Example. For an open set $U \subseteq \mathbb{R}^n$ of Euclidean space expressed in Cartesian coordinates

$$\mathrm{grad}(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}.$$

The divergence and Laplacian may be similarly defined, which we will omit here. We now define differential forms on manifolds. To this end, let

$$\bigwedge_x^k (M^m) \coloneqq \bigwedge^k \left(T_x^* M^m\right).$$

Definition. A *k-form* ω^k on a manifold M^m is a family $\left\{\omega_x^k\right\}$ defining a k-form $\omega_x^k \in \bigwedge_x^k(M^m)$ at each point $x \in M^m$. The differential of a smooth map $f\colon N^n \to M^m$ between two manifolds allows us to pull back k-forms on M^m to k-forms on N^n via the *pullback*

$$(f^*\omega^k)(v_1, \ldots, v_k) = \omega^k(f_*(v_1), \ldots, f_*(v_k)),$$

where $v_1, \ldots, v_k \in T_yN^n$ are tangent vectors and $f_*\colon T_yN^n \to T_{f(y)}M^m$ is the differential of f at this point. This construction can, in particular, be applied to a chart $h\colon V \to M^m$ of the manifold M^m. Hence, for a fixed chart, to every k-form ω^k on M^m corresponds to a k-form $h^*(\omega^k)$ on the open set V in the space $\mathbb{R}^m$ or $\mathbb{H}^m$, respectively. If $y = (y_1, \ldots, y_m)$ are the associated coordinates, then $h^*(\omega^k) = \sum_I \omega_I dy_I$ with multi-indices $I = (i_1 < \cdots < i_k)$ and $dy_I = dy_{i_1} \wedge \cdots \wedge dy_{i_k}$.

Definition. A k-form ω^k on the manifold M^m is called a *differential k-form* or *smooth k-form* if for each chart $h\colon V \to M^m$ the coefficients ω_I of $h^*(\omega^k)$ are smooth functions on $V \subseteq \mathbb{R}^m$. (Hence, we get a module $\Omega^k(M^m)$ of $\mathcal{C}^\infty$-forms of degree k over ring $\mathcal{C}^\infty(M^m)$ of smooth functions on M^m.) We transfer the definition of the exterior derivative to the setting of manifolds:

Definition. For a k-form ω^k on M^m and a chart $h\colon V \to M^m$ we define the $(k+1)$-form, called the exterior derivative,

$$d\omega^k \coloneqq \left(h^{-1}\right)^* \left(d\left(h^*\omega^k\right)\right)$$

on $h(V) \subseteq M^m$. This is independent of the choice of the chart, and hence uniquely defines a global $(k+1)$-form $d\omega^k$ on M^m. Note that $h^*\omega^k$ maps to V, so applying d is well-defined.

All properties of the exterior derivative known from Eucliean space (like $d \cdot d = 0$) are still valid in the situation of a manifold.
Recall: An orientation of a real vector space is the choice of one of two equivalence classes in the set of bases. (Same orientation iff base change matrix has determinant greater than 0.)

Definition. An *orientation* $\mathcal{O}$ of a manifold is a family $\mathcal{O} = \{\mathcal{O}_x\}$ of orientations in all tangent spaces T_xM^m depending continuously on the point x in the following sense: At each point $x \in M^m$ there exists a chart $h\colon V \to M^m$ containing this point such that hte basis $\{h_*(\partial/\partial y_1, \ldots, \partial/\partial y_n)\}$ is compatible with the orientation for every point $y \in V$.

Definition. A manifold is called *orientable* if there exists at least one orientation on it.

Theorem. Let $f_1, \ldots, f_{n-m}\colon U \to \mathbb{R}$ be smooth functions defined on an open subset $U \subseteq \mathbb{R}^n$ and assume that

$$df_1 \wedge \cdots \wedge df_{n-m} \neq 0$$

at each point. Then the manifold

$$M^m = \{x \in U \mid f_1(x) = \cdots = f_{n-m}(x) = 0\}$$

is *orientable.*

Definition. On an oriented submanifold $M^m \subseteq \mathbb{R}^n$, we can define the *volume form* dM^m, a differential form of highest degree. Choose an orthonormal basis $e_1, \ldots, e_m \in T_xM^m$ in the fixed orientation $\mathcal{O}_x$ of any tangent space T_xM^m. Then

$$dM^m(V_1, \ldots, V_m) = \det\left((\langle v_i, e_j\rangle)_{1\leq i,j\leq m}\right).$$

This uniquely determines the form dM^m. (Evaluating dM^m at any orthonormal basis $e'_1, \ldots, e'_m$ in the orientation yiels $dM^m(e'_1, \ldots, e'_m)) = 1$.)

Theorem. An m-dimensional manifold M is orientable iff it carries a nowhere vanishing differential form of degree m. In a chart h, the induced volume form $h^*(dM^m)$ satisfies $h^*(dM^m) = \sqrt{g(y)}dy_1 \wedge \cdots \wedge dy_m$, where $g(y)$ is defined using the Riemannian metric.

Example.
1. The Möbius strip is a non-orientable manifold.
2. The volume form of $\mathbb{R}^n$ in Cartesian coordinates is $d\mathbb{R}^n = dx_1 \wedge \cdots \wedge dx_n$.
3. In polar coordinates on $\mathbb{R}^2 \setminus \{0\}$ the volume form is $d\mathbb{R}^2 = r \cdot dr \wedge d\varphi$.

Remark. The orientation of a manifold induces a unique orientation on its boundary (using the "exterior normal vector field"), but we omit the details here.

We now want to integrate an m-form over an m-dimensional manifold. We restrict to *compact* manifolds for simplicity.

Theorem. Let M^m be a compact manifold. Then there exist smooth functions and charts

$$\varphi_i\colon M^m \to \mathbb{R} \text{ and } h_i\colon V_i \to M^m \ (1 \le i \le l)$$

with

1. $\operatorname{supp}(\varphi_i) \subseteq h_i(V_i)$,
2. the functions φ_i are non-negative and $\sum_{i=1}^{l} \varphi_i(x) = 1$ ("partition of unity").

The partition of unity guarantees that we don't count contributions from overlapping charts twice.

Definition. Let M^m be an oriented, compact manifold. Choose charts h_i and functions φ_i as above. Suppose furthermore that the chart maps $h_i\colon V_i \to M^m$ preserve orientation. Let ω^m be an m-form on M^m. Then

$$h_i^*(\varphi_i \omega^m) := f_i(y) dy_1 \wedge \cdots \wedge dy_m$$

is an m-form on V_i with compact support and we define the *integral*

$$\int_{M^m} \omega^m := \sum_{i=1}^{l} \int_{V_i} h_i^*(\varphi_i \omega^m) = \sum_{i=1}^{l} \int_{V_i} f_i(y) dy_1 \wedge \cdots \wedge dy_m.$$

(This is independent of the chosen partition of unity.)

Definition. The m-dimensional *volume* of a compact and oriented submanifold $M^m \subseteq \mathbb{R}^n$ is

$$\operatorname{vol}(M^m) = \int_V \sqrt{g(y)} \,\mathrm{d}\, y.$$

Example. We can use this formula to show that

$$\operatorname{vol}\left(S^{n-1}(R)\right) = \frac{n}{R} \operatorname{vol}(\mathbb{D}^n(R)),$$

where $S^{n-1}(R)$ and $\mathbb{D}^n(R)$ are the $(n-1)$-dimensional sphere and n-dimensional ball of radius R, respectively.

Theorem (Stokes' Theorem)**.** Let M^k be a compact, oriented manifold, and suppose that the boundary is endowed with the induced orientation. Then, for every $(k-1)$-form ω^{k-1} on M^k,

$$\int_{\partial M^k} \omega^{k-1} = \int_{M^k} \mathrm{d}\,\omega^{k-1}.$$

In particular, if M^k has no boundary, then for every $(k-1)$-form ω^{k-1}

$$\int_{M^k} \mathrm{d}\,\omega^{k-1} = 0.$$

Definition. A connected manifold M^k is called simply connected if every two $\mathcal{C}^1$-curves $c_0, c_1\colon [0,1] \to M^k$ with coinciding initial and end point are homotopic.

Theorem. Every closed 1-form on a simply-connected manifold is exact.

Example. The winding form on $\mathbb{R}^2 \setminus \{0\}$ is closed but not exact. Hence, $\mathbb{R}^2 \setminus \{0\}$ is not simply connected.

Consider two oriented compact manifolds M^k, N^k without boundary and of equal dimensions. Two maps $f_0, f_1\colon M^k \to N^k$ are called homotopic if there exists a smooth map $F\colon M^k \times [0,1] \to N^k$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. Stokes' Theorem implies:

Theorem. Let ω^k be a k-form an N^k and let $f_0, f_1\colon M^k \to N^k$ be homotopic maps. Then

$$\int_{M^k} f_0^*(\omega^k) = \int_{M^k} f_1^*(\omega^k).$$

Theorem. The antipodal map $A\colon S^n \to S^n, A(x) = -x$ is homotopic to the identity iff the dimension is odd.

Theorem (Hedgehog Theorem / Hairy Ball Theorem)**.** A sphere S^{2k} of even dimension does not have a nowhere vanishing, continuous tangent vector field.

Proof. Suppose that there is such a vector field $\mathcal{V}$. Without loss of generality, we may assume that $\mathcal{V}$ is smooth (Stone-Weierstrass Theorem) and that $\mathcal{V}(x)$ has length 1 at each point. We view $\mathcal{V}$ as a function $\mathcal{V}\colon S^n \to \mathbb{R}^n$ satisfying

$$\langle x, \mathcal{V}(x)\rangle = 0, \|\mathcal{V}(x)\| = 1.$$

Define the homotopy $F\colon S^n \times [0,1] \to S^n$ by

$$F(x,t) = \cos(\pi t) \cdot x + \sin(\pi t) \cdot \mathcal{V}(x).$$

The length of $F(x,t)$ is 1 everywhere since x and $\mathcal{V}(x)$ are perpendicular. Moreover, $F(x,0) = x$ and $F(x,1) = -x$, i.e. F is a homotopy between the identity and the antipodal map of S^n. But then n has to be odd. $\square$